$$\underset{m \times n}{A}\,\vec{x} = \vec{0} \qquad \operatorname{rank}(A) = r$$

$\downarrow$ GJ

$$\begin{array}{c} r\{ \\ \\ n-r\{ \end{array} \begin{bmatrix} 1 & \times & \times & 0 & \times & \times & 0 & \times & 0 & \times \\ 0 & & & 1 & \times & \times & 0 & \times & 0 & \times \\ 0 & & & & & & 1 & \times & 0 & \times \\ 0 & & & & & & & & 1 & \times \\ 0 & 0 & & 0 & & 0 & & 0 & & 0 \\ 0 & 0 & & 0 & & 0 & & 0 & & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Big\} r$$

$$\begin{array}{c} r\{ \\ m-r\{ \end{array} \begin{bmatrix} \overbrace{I}^{r} & \overbrace{G}^{n-r} \\ 0 & 0 \end{bmatrix} \underbrace{\overbrace{P\vec{x}}^{y}}_{\text{Permutation}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x = P^T y$$

$$r\{ \begin{bmatrix} \overbrace{I}^{r} & \overbrace{G}^{n-r} \end{bmatrix} \overbrace{\begin{bmatrix} y_d \\ y_f \end{bmatrix}}^{y} \begin{array}{l} \}r \\ \}n-r \end{array} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$I y_d + G y_f = \vec{0}$$

$$\therefore\ y_d = -G y_f$$

$$\vec{y} = \begin{bmatrix} y_d \\ y_f \end{bmatrix} = \begin{bmatrix} -G y_f \\ y_f \end{bmatrix} = \underbrace{\begin{bmatrix} -G \\ I \end{bmatrix}}_{n-r} y_f$$

$$P^T \begin{bmatrix} -G \\ I \end{bmatrix} \rightarrow \text{basis for } N(A)$$

MAB (I)

A Diagonalizable $\Rightarrow A = V\Lambda V^{-1}$.

A symmetric $\Rightarrow A = V\Lambda V^T$

A (Hermitian) $\Rightarrow A = V\Lambda V^*$

Positive Definite مثبت معین

$A \in \mathbb{R}^{n\times n}$, symmetric $A^T = A$

$$\forall x \in \mathbb{R}^n, x \neq \vec{0} \Rightarrow \underbrace{\underset{1\times n}{x^T}\,\underset{n\times n}{A}\,\underset{n\times 1}{x}}_{1\times 1} > 0$$

$$x^T \; A \; x = \square$$

$\forall\, x \neq 0 \quad x^T A x > 0$

$$x^T A y = [x_1 \,—\, x_n] \begin{bmatrix} a_{11} & & \\ & & \\ & & a_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ | \\ y_n \end{bmatrix}$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, x_i\, y_j$$

$$[1 \; 0 \; 0] \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = a_{11} > 0$$

A positive definite

$A \in \mathbb{R}^{n\times n}$ singular $\Rightarrow \exists \vec{x} \in \mathbb{R}, \vec{x} \neq 0 \quad A\vec{x} = \vec{0} \Rightarrow x^T A \vec{x} = 0$

تناقض

A positive $\Rightarrow$ non-singular

$\forall x \in \mathbb{R}^n \quad x^T A x \geq 0 \Rightarrow$ positive semi-definite

$\boxed{x \neq 0 \Rightarrow x^T A x = 0 \Rightarrow \text{singular?}}$

~~positive~~ negative definite $\forall x \neq 0 \quad x^T A x < 0$

A PD $\Rightarrow -A$ ND

negative semi-definite $\exists x \; x^T A x \leq 0$

$A \in \mathbb{C}^{n\times n}$ PD $\rightarrow$ Hermitian $A^* = A$

$\forall x \in \mathbb{C}^n,\ x \neq 0 \quad x^*Ax > 0$

$$\overline{x^*Ax} = x^T\bar{A}\bar{x} = \bar{x}^T\bar{A}^T x = x^*A^*x = x^*Ax$$

$$\Rightarrow \quad (A^* = A \Rightarrow x^*Ax \text{ real})$$

PSD Hermitian

$\forall x \in \mathbb{C}^n \quad x^*Ax \geq 0$

Let $v \in \mathbb{R}^n$ be an eigenvector of $A \in \mathbb{R}^{n\times n}$,

A is PD $\Rightarrow$ $Av = \lambda v$

$$v^TAv > 0 \Rightarrow v^TAv = \lambda v^Tv = \lambda \|v\|^2 > 0$$

$$\Rightarrow \lambda > 0$$

A is PD $\Rightarrow$ all eigenvalues are positive

A is PSD $\Rightarrow$ " " " nonnegative

$u \in \mathbb{R}^n$ $\quad \boxed{uu^T} = A \in \mathbb{R}^{n\times n}$

$$x^TAx = x^T(uu^T)x = \underbrace{x^Tu}_{1\times 1}\,\underbrace{u^Tx}_{1\times 1} = (u^Tx)^2 \geq 0$$

PSD

$n \geq 2 \Rightarrow$ cannot be PD

$\Rightarrow \exists\, x \neq 0 \quad x \perp u \Rightarrow u^Tx = 0 \Rightarrow x^TAx = 0$

$U \in \mathbb{R}^{m\times n}$ $\quad (U^TU)^T = U^TU$ symmetric $\quad$ MA 13 III

$A = U^TU$ $\qquad x^TAx = x^TU^TUx = \underbrace{(Ux)}_{y}{}^T\underbrace{(Ux)}_{y\in\mathbb{R}^m} = y^Ty = \|y\|^2 \geq 0$

$$x^TAx = x^TU^TUx = \|Ux\|^2 \geq 0$$

U^TU is PSD

when is U^TU positive definite?

U^TU PD $\Rightarrow \forall \vec{x} \neq 0 \quad x^TU^TUx = \|Ux\|^2 > 0$

$\Rightarrow Ux \neq \vec{0}$ for all $x \neq 0$.

columns of U are linearly independent $\Rightarrow$ U has full column rank.

$$U = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix}$$

UU^T PD $\Rightarrow$ U is full-row-rank..

$A \in \mathbb{R}^{n\times n}$ is PSD $\Rightarrow \exists\, U \in \mathbb{R}^{n\times n}$ $\underline{A = U^TU}$

$\exists\, V \in \mathbb{R}^{n\times n}$ $A = VV^T$

$\text{rank}(UU^T) = \text{rank}(U)$ $\qquad C(UU^T) = C(U)$

$\text{rank}(AB) \leq \underset{\text{rank}(A)}{\text{rank}(B)}$ $\qquad$ Case 1: Assume U is full-col-rank

Scanned with CamScanner

(IV)

A PSD $\Rightarrow A = UU^T$

~~Anny~~ Let H be any orthogonal matrix ($HH^T = I$)

$$UU^T = UIU^T = UHH^TU^T = (UH)(UH)^T$$

A is PD $\Rightarrow$

~~A~~ There exist a unique $P \in \mathbb{R}^{n\times n}$ P is PD

$$A = PP^T = PP$$

P is called the square root of A.

$$A^{1/2}$$

Choleskey Decomposition

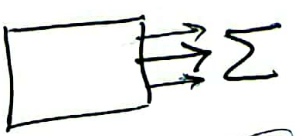
A is PSD $\Rightarrow \boxed{A} = \boxed{L\,L^T}$

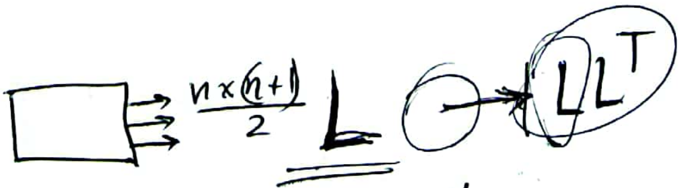
lower triangular

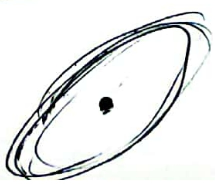
$$PA = LU \qquad \boxed{Ax=b} \Rightarrow P\,L\underbrace{Ux}_{y} = b$$

$$Ly = b \Rightarrow y = \checkmark \Rightarrow Ux = y \Rightarrow x = \checkmark$$

$$AA^{-1} = I \Rightarrow AX = I \Rightarrow A\,[x_1\ x_2 \cdots x_n] = [e_1\ e_2 - e_n]$$







VAE

$x_1 \quad x_2 \text{ ——— } x_n$

$y_1 \quad y_2 \text{ ——— } y_n$

correlation .

$$\text{corr}(x,y) = \sum_{i=1}^{1} x_i y_i$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}, \dots, \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix}$$

$\downarrow \vec{V}_1 \quad \downarrow \vec{V}_2 \quad \downarrow \vec{V}_n$

$$C = \begin{bmatrix} \text{corr}(x,x) & \text{corr}(x,y) & \text{corr}(x,z) \\ \text{corr}(y,x) & \text{corr}(y,y) & \text{corr}(y,z) \\ \text{corr}(z,x) & \text{corr}(z,y) & \text{corr}(z,z) \end{bmatrix}$$

$C \downarrow$ correlation matrix

C is always PSD

$$C = \sum_{i=1}^{n} \begin{bmatrix} x_i^2 & x_i y_i & x_i z_i \\ x_i y_i & y_i^2 & y_i z_i \\ x_i z_i & y_i z_i & z_i^2 \end{bmatrix} = \sum_{i=1}^{n} \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} \begin{bmatrix} x_i & y_i & z_i \end{bmatrix}$$

$$= \sum_{i=1}^{n} V_i V_i^T$$

$$x^T C x = x^T \left(\sum V_i V_i^T \right) x$$
$$= \sum x^T V_i V_i^T x$$
$$= \sum (x^T V_i)^2 \geq 0$$

$$\sum_{i=1}^{n} V_i V_i^T = \underbrace{\begin{bmatrix} V_1 & V_2 & \text{—} & V_n \end{bmatrix}}_{V} \begin{bmatrix} - V_1^T - \\ - V_2^T - \\ \vdots \\ - V_n^T - \end{bmatrix} = V V^T$$

$$V = \begin{bmatrix} x_1 & x_2 & \text{——} & x_n \\ y_1 & y_2 & \text{——} & y_n \\ z_1 & z_2 & \text{——} & z_n \end{bmatrix}$$

$\bar{V}_i =$

$$\mu = 1/n \sum_{i=1}^{n} V_i$$

$$\Sigma = \sum_{i=1}^{n} (V_i - \mu)\overbrace{(V_i - \mu)}^{\bar{V}_i}{}^T$$

$\Sigma \downarrow$ covariance matrix

$$\bar{V}_i = V_i - \mu$$

$$\bar{V} = \begin{bmatrix} \bar{V}_1 & \bar{V}_2 & \text{—} & \bar{V}_n \end{bmatrix}$$

$$\Sigma = \bar{V} \bar{V}^T$$

VI

$\Sigma = \begin{bmatrix} 2 & +1 \\ +1 & 2 \end{bmatrix}$ $\quad \Sigma = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ $\quad \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\underline{A \text{ PD}} \Rightarrow \underline{A^{-1} \text{ is PD}}$

$\forall x \neq 0 \quad x^T A^{-1} x > 0$

LDL^T Decomposition

$\underline{A = LDL^T}$ $\quad L = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ x & x & 1 \end{bmatrix}$

D: diagonal

Least squares $\quad Ax = b$

$A \in \mathbb{R}^{m \times n} \quad m > n$

$\underbrace{(A^T A)}_{\text{PD}} x = \underline{A^T b} \Rightarrow x = (A^T A)^{-1} A^T b$