MA 15

I

$$A = \underset{m\times m}{U}\ \underset{m\times n}{\Sigma}\ \underset{n\times n}{V^T} \quad \text{full}$$

U: $m\times m$ → orthogonal

Σ: $m\times n$ → diagonal

V^T: $n\times n$ → orthogonal

$$A = U\Sigma V^T \qquad q = \min(m, n) \quad \text{skinny}$$

A: $m\times n$, U: $m\times q$, Σ: $q\times q$, V^T: $q\times n$

V: $n\times q$

$U = [u_1 — u_m]$ left sin vecto

$V = [v_1 — v_m]$ right sing vect

$$\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_q \end{bmatrix}$$

singular value

$$\overset{m\times n}{A} = U\Sigma V^T$$

$$\underbrace{A^T A}_{n\times n} = (U\Sigma V^T)^T (U\Sigma V^T) = V\Sigma^T \underbrace{U^T U}_{I} \Sigma V^T$$

متماثل

$$= V \underbrace{\underset{n\times m}{\Sigma^T}\ \underset{m\times n}{\Sigma}}_{n\times n} V^T$$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \Sigma^T\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \sigma_1^2 & & \emptyset \\ & \sigma_2^2 & \\ \emptyset & & \sigma_3^2 \end{bmatrix}}_{\Lambda}$$

$$A^T A = V\Lambda V^T \qquad \Lambda = \Sigma^T\Sigma$$

MA15 II

$$X = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix} \quad \bar{X} = \begin{bmatrix} x_1-\mu, & x_2-\mu, & \cdots, & x_n-\mu \end{bmatrix}$$

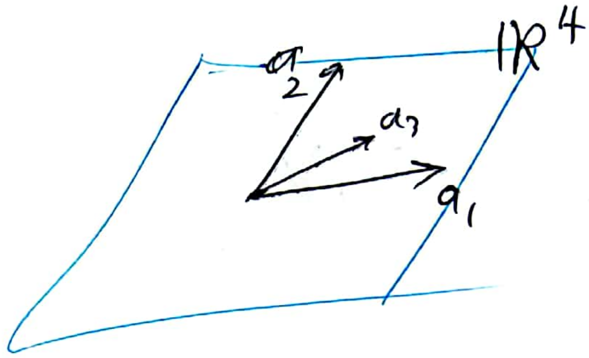
$$\mathrm{Cov} = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)(x_i-\mu)^T = \frac{1}{n}\bar{X}\bar{X}^T$$

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix}$$

4×3

$\mathrm{rank}(A) = 2$

$\vec{a}_1, \vec{a}_2, \vec{a}_3 \in \mathbb{R}^4$



$$\mathrm{rank}(A + \mathrm{noise}(4,3)) = 3$$

$$A = U\Sigma V^T = U\begin{bmatrix} 10.2 & 0 & 0 \\ 0 & 7.1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^T$$

$$A + \underset{\text{noise}}{\mathcal{E}} = U'\Sigma' V'^T = U'\begin{bmatrix} 10.2001 & 0 & 0 \\ 0 & 7.09998 & 0 \\ 0 & 0 & 0.00001 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A = \sigma_1 \vec{u}_1 v_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \underline{\underline{\sigma_3}}\, \vec{u}_3 v_3^T$$

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vector norm $u, v \in \mathbb{R}^n$

$\|\alpha \vec{v}\| = |\alpha|\, \|\vec{v}\|$

$\|v\| \geq 0$

$\|v\| = 0 \iff \vec{v} = \vec{0}$

$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

matrix norm $A, B \in \mathbb{R}^{m\times n}$

$\|\alpha A\| = |\alpha|\, \|A\|$

$\|A\| \geq 0$

$\|A\| = 0 \iff A = 0_{m\times n}$

$\|A + B\| \leq \|A\| + \|B\|$

$\|AB\| \leq \|A\|\,\|B\|$ submultiplicative norm

Operator Norm

$$\underset{\mathbb{R}^m}{y} = \underset{m\times n}{A}\, x_{\in \mathbb{R}^n}$$

$$\max_x \left\{ \|Ax\|_{norm1} \;\middle|\; \|x\|_{norm2} = 1 \right\}$$

$$\max_x \frac{\|Ax\|_{norm1}}{\|x\|_{norm2}}$$

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spectral norm

$$\rho = \max_x \frac{\|Ax\|_2}{\|x\|_2} = \sigma_1 = \sigma_{max}$$

A (symetric) positive definite

$$A = V\Lambda V^T = V \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} V^T$$

$$\lambda_1, \lambda_2 - \lambda_n > 0$$

$$A^{1/2} = V \begin{bmatrix} \lambda_1^{1/2} & & & \\ & \lambda_2^{1/2} & & \\ & & \ddots & \\ & & & \lambda_n^{1/2} \end{bmatrix} V^T$$

$$A^{1/2}A^{1/2} = V \begin{bmatrix} \diagdown \end{bmatrix} \underbrace{V^T V}_{I} \begin{bmatrix} \diagdown \end{bmatrix} V^T = V \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} V^T$$

$$\|A\|_* = \mathrm{trace}\left((A^TA)^{1/2}\right) =$$

$$\underset{\in \mathbb{R}^{m\times n}}{A = U\Sigma V^T} \qquad A^TA = V\Sigma^T\Sigma V^T = V \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_n^2 \end{bmatrix} V^T$$

$$(A^TA)^{1/2} = V \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix} V^T$$

$A \in \mathbb{R}^{n \times n}$ Symmetric

$x \in \mathbb{R}^n$, $x^T A x \in \mathbb{R}$

$$\underset{1\times n}{\begin{bmatrix} x^T \end{bmatrix}} \underset{n \times n}{\begin{bmatrix} A \end{bmatrix}} \underset{n \times 1}{\begin{bmatrix} x \end{bmatrix}} \in \mathbb{R}$$

$$\underset{x \in \mathbb{R}^n}{\operatorname{argmax}} \quad x^T A x \quad \text{s.t.} \quad \|x\|_2 = 1$$

$$\max_x \quad x^T A x \quad \text{s.t.} \quad \|x\| = 1$$

$$\|x\|^2 = 1$$

$$x^T x = 1$$

$$\max_x \frac{x^T A x}{x^T x} \qquad A \text{ symmetric}$$

$$\min_x \frac{x^T A x}{x^T x}$$

$$A \text{ symmetric} \Rightarrow A = V \Lambda V^{-1} = V \Lambda V^T$$

$$A = \begin{bmatrix} v_1 - v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \emptyset \\ & \ddots & \\ \emptyset & & \lambda_n \end{bmatrix} \begin{bmatrix} v_1^T \\ | \\ v_n^T \end{bmatrix} = \cancel{I} \qquad \|v_i\| = 1$$

$$A = \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T + \dots + \lambda_n v_n v_n^T \qquad \|x\| = 1$$

$$Ax = \lambda_1 v_1 (v_1^T x) + \lambda v_2 (v_2^T x) + \dots + \lambda_n v_n (v_n^T x)$$

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$$

$$x^T A x = \lambda_1 \|x^T v_1\|^2 + \lambda_2 \|x^T v_2\|^2 + \dots + \lambda_n \|x^T v_n\|^2$$

$$\max \|Ax\|_2$$

$A \in \mathbb{R}^{m\times n}$

$$\max_x \|A\vec{x}\|_2 \quad s.t. \quad \|x\|_2 = 1$$

$$\min_x \|A\vec{x}\|_2 \quad \text{subject to} \quad \|x\|_2 = 1$$

$$\|Ax\|^2 = \langle Ax, Ax\rangle = (Ax)^T(Ax) = x^T A^T A x$$

$m \geq n$

$$\underset{m\times n}{A} = \begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix} \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \ddots \; \sigma_n \\ 0 & 0\; 0\; 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix}$$

$$\max_x \frac{\|Ax\|}{\|x\|} = \sigma_1 = \sigma_{max} \;/\; \text{argmax} = \vec{v_1}$$

$$\min_x \frac{\|Ax\|}{\|x\|} = \sigma_n = \sigma_{min} \;/\; \text{argmin} = v_n^T$$

$$\underset{m\times n}{\begin{bmatrix} A \end{bmatrix}} \vec{x} = \vec{0} \qquad m \geq n$$

جواب یکتا $\quad \underline{\text{rank}(A)} = \underline{n-1}$

$\parallel$ $\dim(N(A)) = n - \text{rank}(A) = 1$

In practice $\text{rank}(A) = n$ $\quad \dim(N(A)) = 0$, $N(A) = \{\vec{0}\}$

but A is near_low_rank

$$\underline{\|Ax\| > 0} \qquad x \neq 0$$

MA15 IV

$$\frac{Ax \simeq 0}{\|Ax\| \geq 0}$$

$$x = \underset{x}{\operatorname{argmin}} \|Ax\| \text{ s.t. } \|x\| = 1$$

$\cancel{U, \Sigma, V^T}$

$$A = U \Sigma V^T$$

$$= U \Sigma \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix}$$

$$x^* = \operatorname{argmin} \|Ax\| \text{ s.t. } \|x\| = 1 = \underline{\underline{v_n^T}}$$