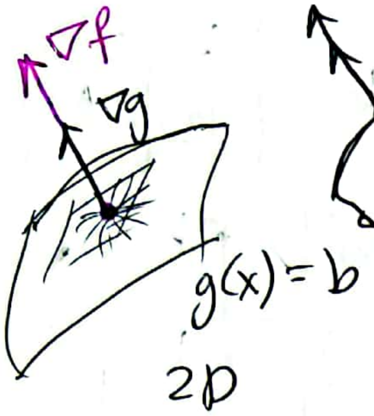
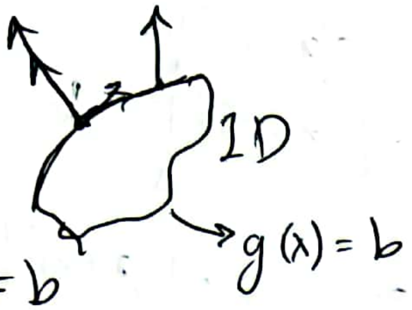
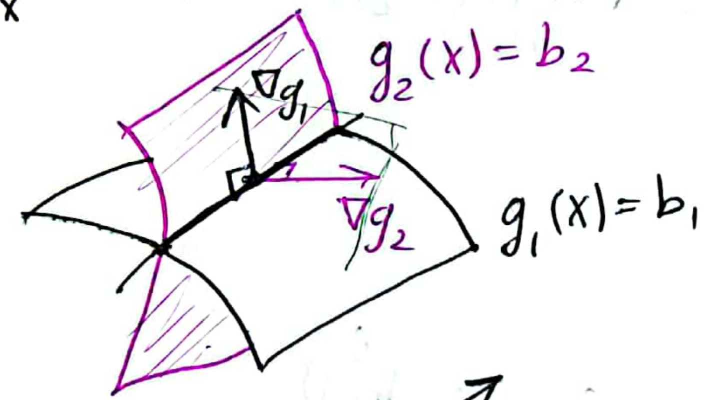
$$\min_{x} f(x) \quad \text{s.t.} \quad g(x) = b$$



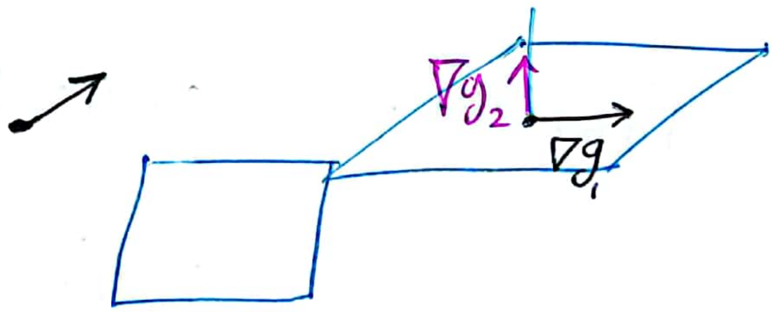


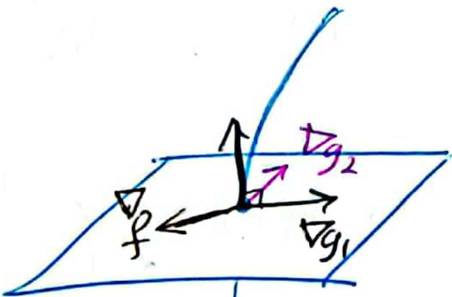
$$\nabla f(x) + \lambda \nabla g(x) = 0$$

$$\min_{x} f(x) \quad \text{s.t.} \quad g_1(x) = b_1 \ , \ g_2(x) = b_2$$



$$\nabla f \in \operatorname{span}(\nabla g_1, \nabla g_2)$$





$$\nabla f \in \operatorname{span}(\nabla g_1, \nabla g_2)$$

find x such that $\nabla f(x) = \lambda_1 \nabla g_1(x) + \lambda_2 \nabla g_2(x)$

$$\boxed{\nabla f + \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 = 0}$$

$$\mathcal{L}(x, \lambda_1, \lambda_2) = f(x) + \lambda_1 (g_1(x) - b_1) + \lambda_2 (g_2(x) - b_2)$$

$$\mathcal{L}(x,\lambda_1,\lambda_2) = f(x) + \lambda_1(g_1(x)-b_1) + \lambda_2(g_2(x)-b_2)$$

$g : \mathbb{R}^n \to \mathbb{R}^2 \quad g(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix} \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad \vec{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$

$$\mathcal{L}(x,\lambda_1,\lambda_2) = f(x) + \underbrace{[\lambda_1, \lambda_2]}_{1\times 2}\underbrace{\left(\begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\right)}_{2\times 1}$$

$$\mathcal{L}(x,\vec{\lambda}) = f(x) + \vec{\lambda}^T(g(x)-\vec{b})$$

the vector of lagrange multipliers

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m constrairst

$$\min_x f(x) \quad \text{s.t.} \quad g(x) = \emptyset$$

$f : \mathbb{R}^n \to \mathbb{R} \qquad g : \mathbb{R}^n \to \mathbb{R}^m \quad \vec{b} \in \mathbb{R}^m$

$$\mathcal{L}(x,\lambda) = f(x) + \lambda^T g(x) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$$

$x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m$

$$\nabla_x \mathcal{L}(x,\lambda) = \nabla_x f(x) + \sum_{i=1}^{m} \lambda_i \nabla_x g_i(x)$$

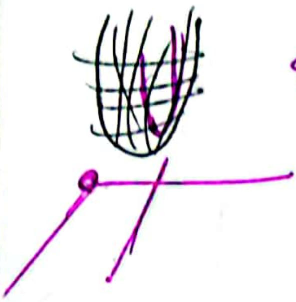
$$= \nabla f(x) + \underbrace{\begin{bmatrix} \nabla_x g_1(x) & \nabla g_2 & \cdots & \nabla g_m \end{bmatrix}}_{n\times m} \begin{bmatrix} \lambda_1 \\ | \\ \lambda_m \end{bmatrix}$$

solve for x, λ $\leftarrow$ $= \begin{cases} \nabla f(x) + J_g^T \lambda = \emptyset \\ g(x) = \emptyset \end{cases}$

$$f(x) = \frac{1}{2} x^T H x + u^T x + c$$

$$f: \mathbb{R}^n \to \mathbb{R}$$

symmetric & positive definite



m Linear constraints

$$\underbrace{\vec{A}}_{m\times n} x = \vec{b}$$

$$g(x) = Ax$$

$$\underbrace{\begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix}}_{A} x = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{b}$$

min $f(x)$ subject to $g(x) = \vec{b}$

$$\min_x \frac{1}{2} x^T H x + u^T x + c \quad \text{s.t.} \quad \underbrace{A}_{m\times n} \underbrace{x}_{n\times 1} = \underbrace{\vec{b}}_{m\times 1}$$

$$\mathcal{L}(x,\lambda) = \frac{1}{2} x^T H x + u^T x + c + \lambda^T (Ax - b)$$

$$= \frac{1}{2} x^T H x + u^T x + c + \underbrace{\underbrace{\lambda^T}_{1\times m} \underbrace{A}_{m\times n}}_{1\times n} x - \lambda^T b$$

$$\nabla_x \mathcal{L} = \frac{\partial L}{\partial x} = H\underline{x} + u + A^T \underline{\underline{\lambda}}$$

$$\begin{cases} Hx + u + A^T\lambda = \vec{0} \in \mathbb{R}^n \\ \underbrace{A}_{m\times n} x = b \in \mathbb{R}^m \end{cases}$$

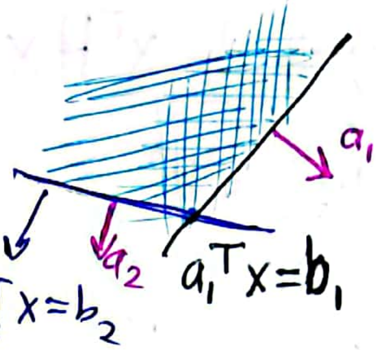
$$\underbrace{\begin{array}{c} \\ n\{ \\ m\{ \end{array}\begin{bmatrix} \overbrace{H}^{n} & \overbrace{\underbrace{A^T}_{n\times m}}^{m} \\ \underbrace{A}_{m\times n} & 0_{m\times m} \end{bmatrix}}_{B} \begin{bmatrix} x \\ \lambda \end{bmatrix}\begin{array}{c} \}n \\ \}m \end{array} = \underbrace{\begin{bmatrix} -u \\ b \end{bmatrix}}_{d}\begin{array}{c} \}n \\ \}m \end{array}$$

```
x_lmb = np.linalg.solve(B,d)
x = x_lmb[:n]
lmb = x_lmb[n:]
```

$$\min_x f(x) \quad \text{s.t.} \quad a_1^T x \leq b_1, \quad a_2^T x \leq b_2$$

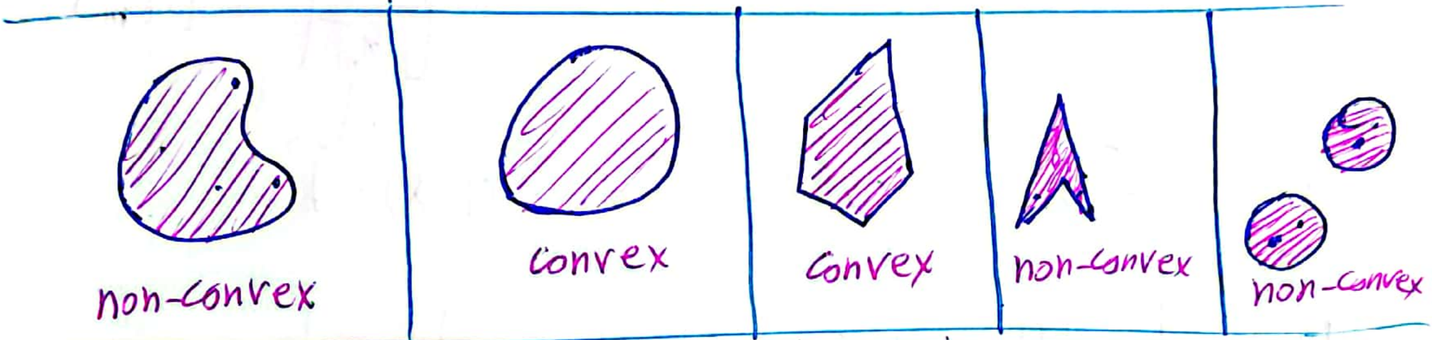
$f: \mathbb{R}^2 \to \mathbb{R}$

$x \in \mathbb{R}^2$



Convex Set

مجموعه محدب



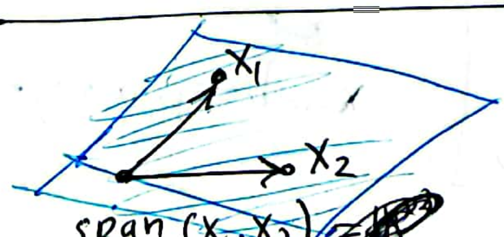
$x_1, x_2 \in \mathbb{R}^n$

مجموعه نقاط روی این پاره خط $= L = \{\alpha x_1 + (1-\alpha) x_2 \mid 0 \leq \alpha \leq 1\}$

$= \{\alpha x_1 + \beta x_2 \mid \alpha + \beta = 1, \alpha \geq 0, \beta \geq 0\}$

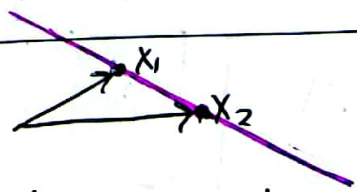
A set $S \subseteq V$ for a vector V is convex if for any pair of points $x_1, x_2 \in S$ and any $0 \leq \alpha \leq 1$ we have

$$\alpha x_1 + (1-\alpha) x_2 \in S$$



$\text{span}(x_1, x_2)$

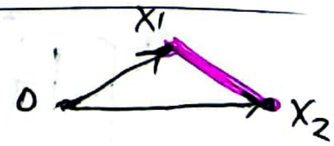
$= \{\alpha x_1 + \beta x_2\}$



$= \{\alpha x_1 + \beta x_2 \mid \alpha + \beta = 1\}$

$= \{\alpha x_1 + (1-\alpha) x_2\}$

affine span



$= \{\alpha x_1 + \beta x_2 \mid \alpha + \beta = 1, \alpha \geq 0, \beta \geq 0\}$

$x_1, x_2 \in V$

$\alpha x_1 + \beta x_2$ linear combination

$\alpha x_1 + \beta x_2$, $\alpha + \beta = 1$ affine combination

$\alpha x_1 + \beta x_2$, $\alpha + \beta = 1$, $\alpha \geq 0, \beta \geq 0$ convex combination

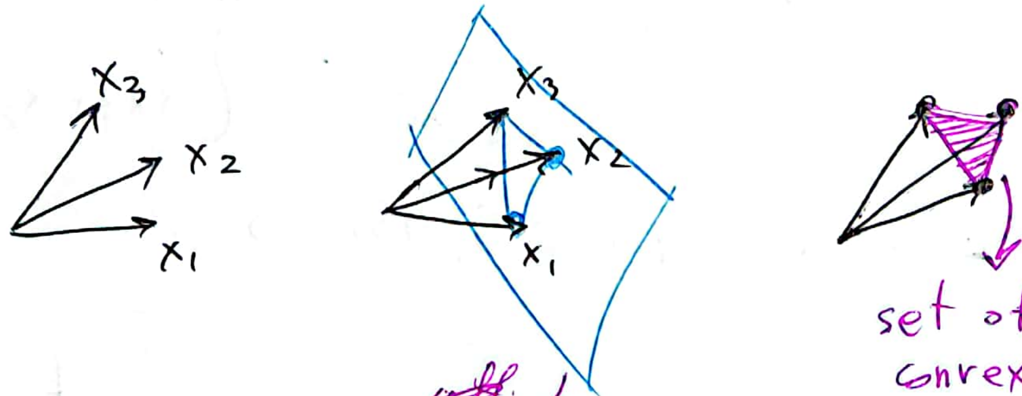
---

$\sum_{i=1}^{n} \alpha_i \cdot \vec{x}_i$ linear comb.

$\sum_{i=1}^{n} \alpha_i \cdot \vec{x}_i$ $\sum_{i=1}^{n} \alpha_i = 1$ ~~aff~~ affine comb.

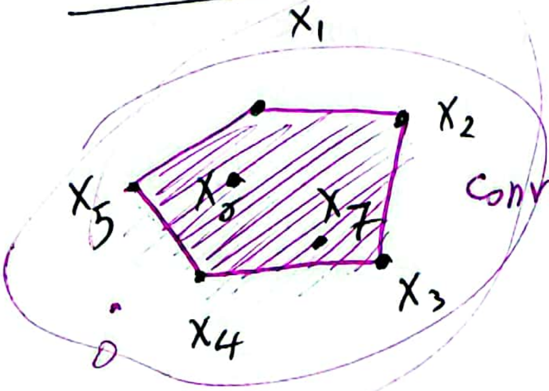
$\sum_{i=1}^{n} \alpha_i \cdot \vec{x}_i$ $(\sum \alpha_i = 1, \ \alpha_i \geq 0)$

↳ a convex comb. of $x_1, x_2, \ldots, x_n$



~~aff~~ set of affine combination

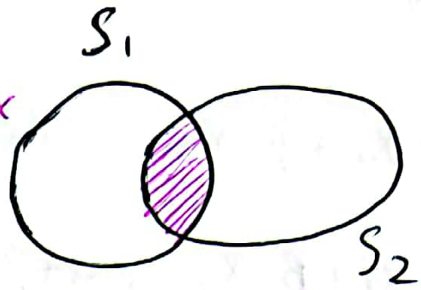
set of convex combinations (convex-hull)



convex hull $= \left\{ \sum_{i=1}^{n} \alpha_i \cdot \vec{x}_i \ \middle| \ \sum_{i=1}^{n} \alpha_i = 1, \ \alpha_i \geq 0 \right\}$

S_1, S_2 convex

$S_1 \cup S_2$ may be non-convex



$S_1 \cap S_2$

S_1, S_2 convex

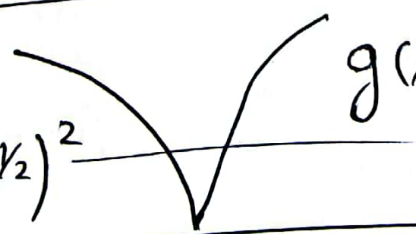
prove that $S = S_1 \cap S_2$ is convex

Let $x_1, x_2 \in S \Rightarrow \begin{cases} x_1 \in S_1 \cap S_2 \Rightarrow \boxed{x_1 \in S_1}, x_1 \in S_2 \\ x_2 \in S_1 \cap S_2 \Rightarrow \boxed{x_2 \in S_1}, x_2 \in S_2 \end{cases}$

for any α, β
$\alpha + \beta = 1$
$\alpha, \beta \geq 0$
$\Rightarrow \begin{cases} \alpha x_1 + \beta x_2 \in S_1 \\ \alpha x_1 + \beta x_2 \in S_2 \end{cases}$

$\Rightarrow \alpha x_1 + \beta x_2 \in S_1 \cap S_2 = S$

$g(x) = \left(\sum_{i=1}^{n} |x_i|^{1/2}\right)^2$



$g(x)$

prove that

$S = \{x \mid g(x) \leq 1\}$

is convex

$x_1 \in S \Rightarrow g(x_1) \leq 1$

$x_2 \in S \Rightarrow g(x_2) \leq 1$

مجموعه بسته

MA24

VII

closed set

S is a closed set if every convergent sequence $x_1, x_2, x_3, \dots \in S$ converges to some point within S.

0 ... 1

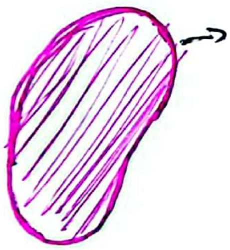
colsed

$[0, 1]$

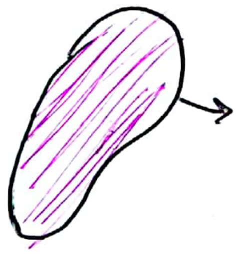
not closed

$[0, 1)$

$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \dots \to 1$



$\to S$



$\to \partial S$

$S \setminus \partial S$

Closed & convex

supporting hypeplane