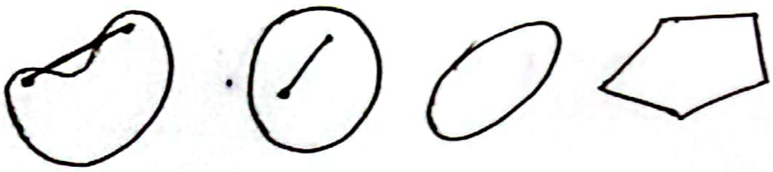
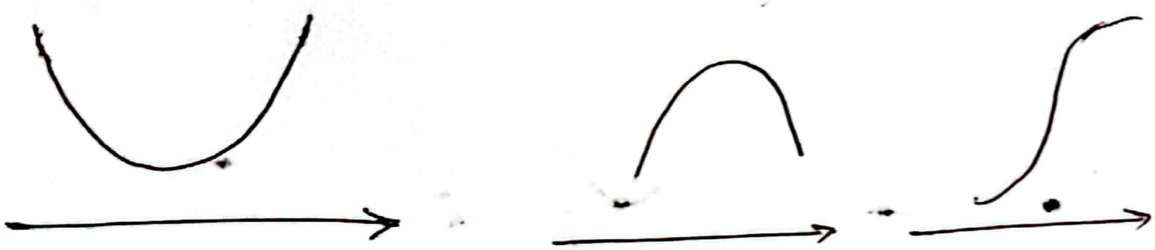
Convex Set



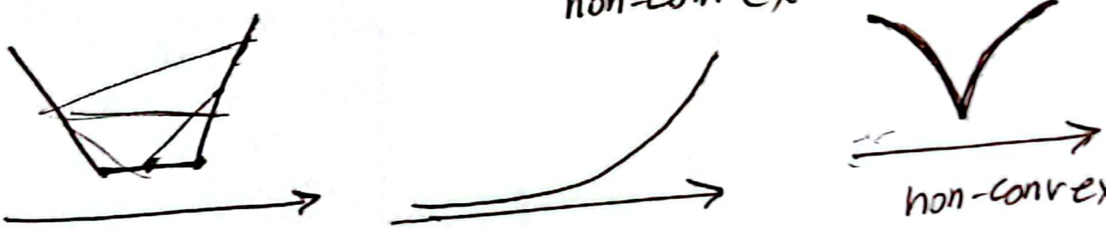
Convex Functions



convex

concave
non-convex

non-convex

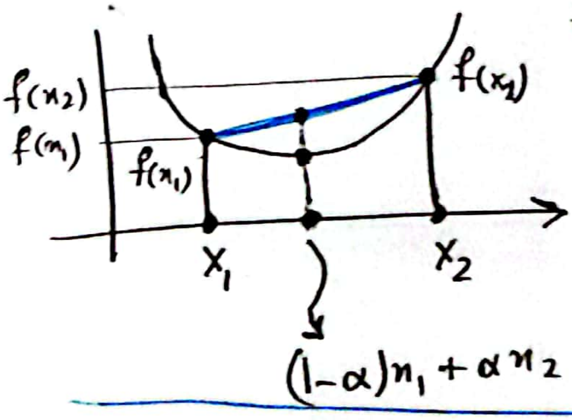


convex

convex

non-convex

$f: \mathbb{R} \to \mathbb{R}$ $f: D \to \mathbb{R}$



for ~~all~~ $0 \leqslant \alpha \leqslant 1$ or $0 < \alpha < 1$

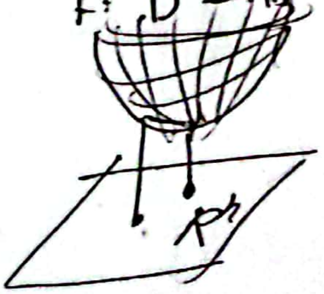
$$f((1-\alpha)x_1 + \alpha x_2) \leqslant (1-\alpha) f(x_1) + \alpha f(x_2)$$

f is called a convex function.

تابع محدب

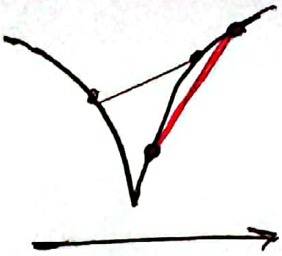
$f: \mathbb{R}^n \to \mathbb{R}$

$f: D \to \mathbb{R}$

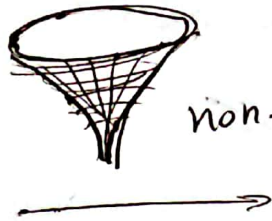


for any $\vec{x}_1, \vec{x}_2 \in D$ & any $0 < \alpha < 1$

$$f((1-\alpha)\vec{x}_1 + \alpha \vec{x}_2) \leqslant (1-\alpha) f(x_1) + \alpha f(x_2)$$

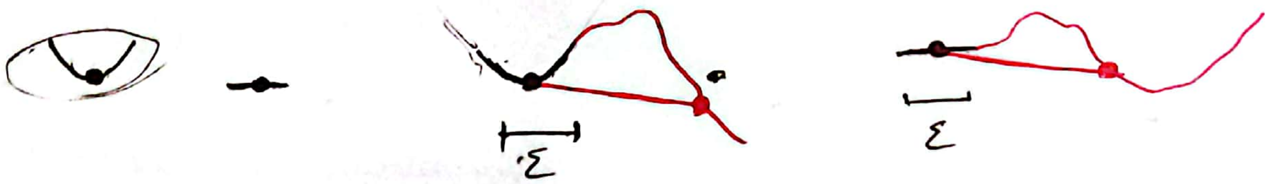


non-convex

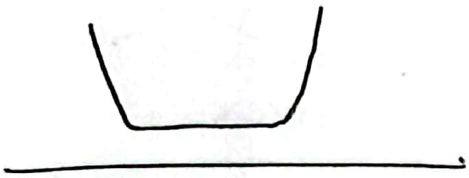


non-convex
(quasi-convex)

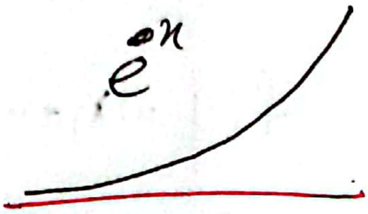
F convex → any local minimum is a global minimum.

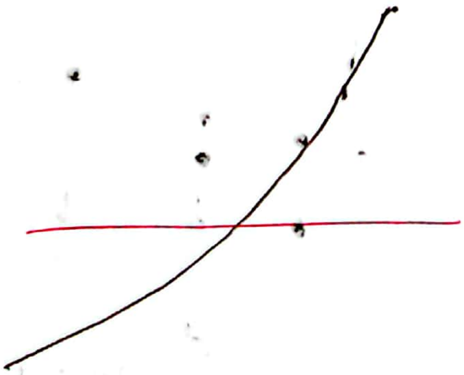


A corex function can have many global minimums.

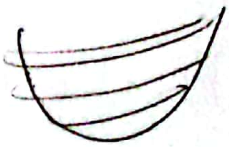


A convex function ~~can~~ may have no ~~to~~ minimum

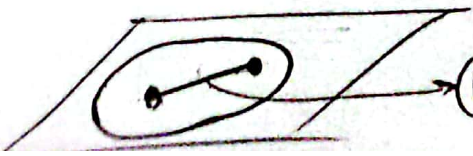




$f: D \to \mathbb{R}$ D must be convex for convexity to be well-defined



a be $\in D$



$(1-\alpha)x_1 + \alpha x_2$ must exist for any $x_1, x_2 \in D$

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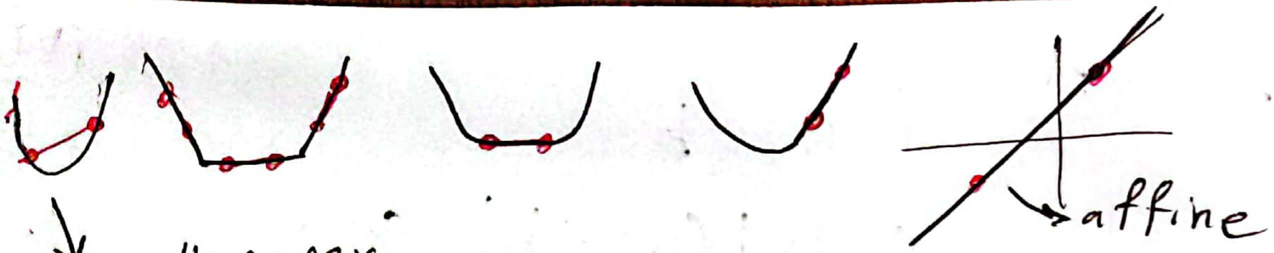
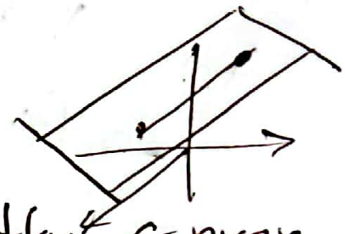


[illegible]

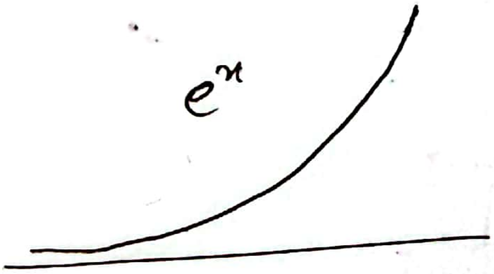


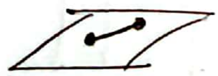
- $f: D \to \mathbb{R}$ is strictly convex

  for all $x_1, x_2 \in D$, any $\alpha \in (0,1)$

  $$f((1-\alpha)x_1 + \alpha x_2) < (1-\alpha) f(x_1) + \alpha f(x_2)$$

~~the~~ (global) minimum is unique or non-existant





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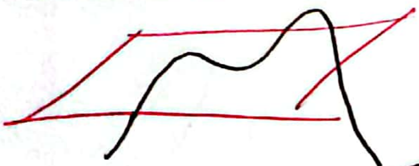
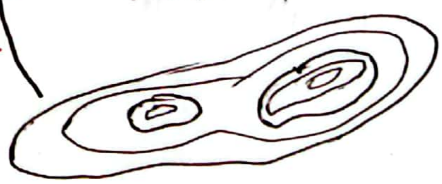
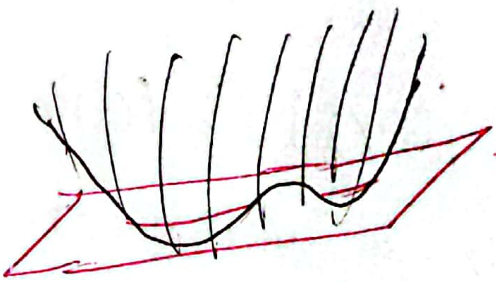
level set



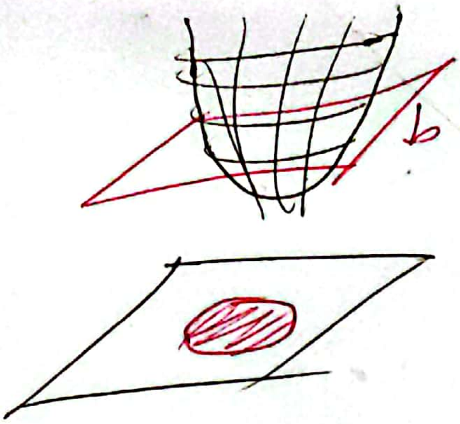
$f(x) = b$





$\{x \mid f(x) = b\}$ level set

$\{x \mid f(x) \leq b\}$ sublevel set



$\cancel{\{f(x) < b}$

$S_b = \{x \mid f(x) \leq b\}$ convex

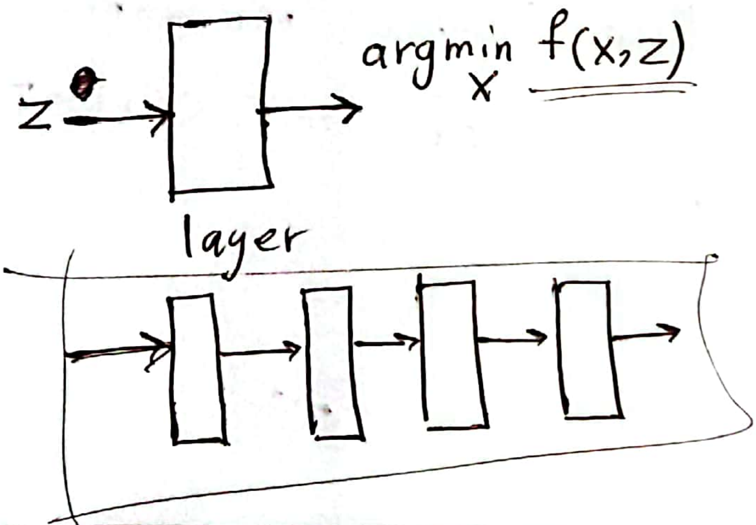
$x_1, x_2 \in S_b \overset{?}{\Longrightarrow} (1-\alpha)x_1 + \alpha x_2 \in S_b$ $\quad \alpha \in (0,1)$

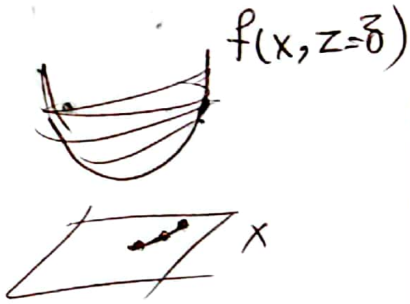
$\Rightarrow f(x_1) \leq b \;, f(x_2) \leq b$

f convex. $\underbrace{f((1-\alpha)x_1 + \alpha x_2)} \leq (1-\alpha) f(x_1) + \alpha f(x_2)$

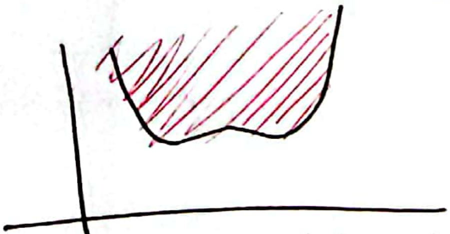
$\boxed{\begin{array}{l} \alpha > 0 \\ 1-\alpha > 0 \end{array}}$ $\quad \leq (1-a) b + \alpha b = \underline{\underline{b}}$

$f((1-a)x_1 + \alpha x_2) \leq b \Longrightarrow (1-\alpha)x_1 + \alpha x_2 \in S_b$



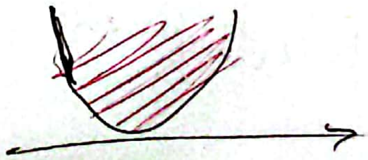


$f: \mathbb{R} \to \mathbb{R}$



$f: \mathbb{R}^n \longrightarrow \mathbb{R}$ $\qquad$ epigraph$(f) = \{(x,y) \mid y \geq f(x)\}$

f: convex $\Rightarrow$ epigraph(f) is a convex set.



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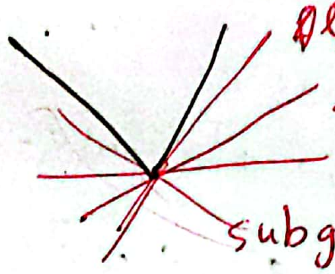
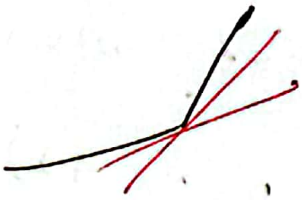
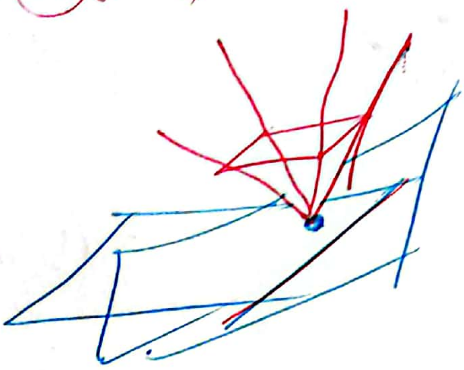


$\ell_1(x) = \ell_1(x_0) + 2(x-x_0)$

$\ell_2(x) = \ell_2(x_0) + 0.5(x-x_0)$

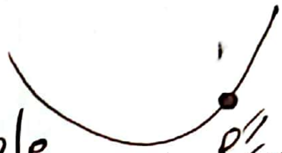




$f:\mathbb{R}^n \longrightarrow \mathbb{R}$

$$\text{subgrad}(f)\Big|_{x_0} = \left\{ m \in \mathbb{R}^n \mid f(x) \geq f(x_0) + m^T (x - x_0) \right\}$$

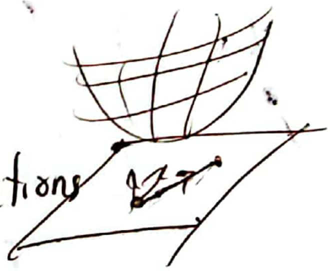
$f: \mathbb{R} \to \mathbb{R}$ convex & twice differentiable



$f''(x) \geq 0$

$f: \mathbb{R} \to \mathbb{R}$ strictly convex $f''(x) > 0$

$f: \mathbb{R}^n \longrightarrow \mathbb{R}$



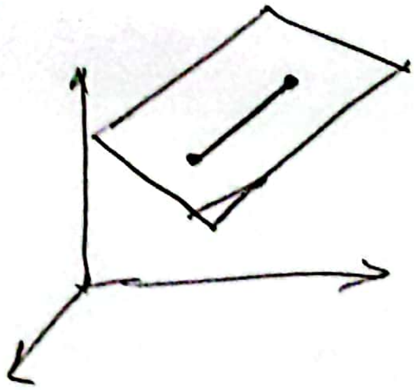
curvature > 0 in all directions

Hessian

Curvature in direction u at x = $\boxed{u^T H u} \geq 0 \quad \forall u$

f convex & twice differentiable $\Longrightarrow$ H positive semi-definite

f strictly convex $\Longrightarrow$ H positive definite



$f(x) = a^T x + b$

$$f((1-\alpha)x_1 + \alpha x_2) = a^T((1-\alpha)x_1 + \alpha x_2) + b$$

$$= (1-\alpha) a^T x_1 + \alpha a^T x_2 + b$$

$$= \underline{(1-\alpha) a^T x_1} + \alpha a^T x_2 + \underline{(1-\alpha) b} + \alpha b$$

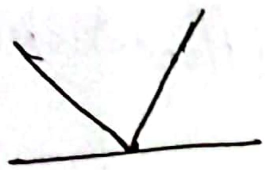
$$= (1-\alpha)(a^T x_1 + b) + \alpha (a^T x_2 + b)$$

$$= (1-\alpha) f(x_1) + \alpha f(x_2)$$

affine function $\Rightarrow$ both convex & concave

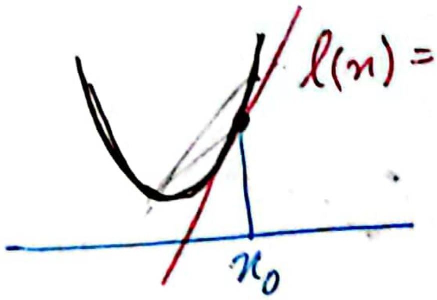
$f(x) = |x|$

$f: \mathbb{R} \to \mathbb{R}$



$f(x) = \|x\|_1$

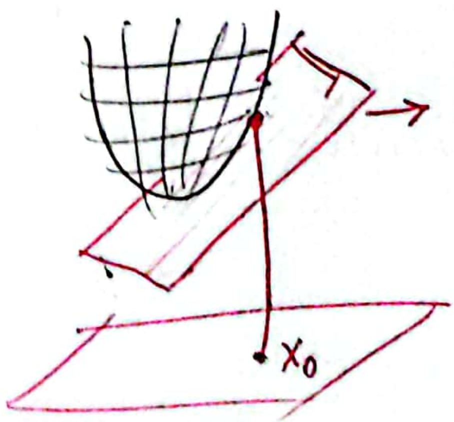
$f: \mathbb{R}^n \to \mathbb{R}$



$$l(x) = f(x_0) + \underset{\substack{\downarrow \\ f'(x_0)}}{m}(x - x_0)$$

f: convex & differentiable

$$f(x_0) + \underset{\substack{\downarrow \\ f'(x_0)}}{m}(x - x_0) \leq f(x)$$



$$l(x) = f(x_0) + \nabla f(x_0)^T (x - x_0)$$

for all $x_0, x \in \mathbb{R}^n$.

$$f(x) \geq f(x_0) + \nabla f(x_0)^T (x - x_0)$$

$h(x) = f(g(x))$ $\quad f \circ g(x)$ convex? $\quad$ MA 25 VII

f: convex
g: convex

$$\boxed{f(g((1-\alpha)x_1+\alpha x_2)) \leq (1-\alpha)f(g(x_1)) + \alpha f(g(x_2))} \ ?$$

g convex $\quad g((1-\alpha)x_1 + \alpha x_2) \leq (1-\alpha)g(x_1) + \alpha g(x_2)$

f convex $\quad f((1-\alpha)g(x_1) + \alpha g(x_2)) \leq \cancel{\alpha f}\ (1-\alpha)f(g(x_1)) + \alpha f(g(x_2))$

need $\underbrace{f(g((1-\alpha)x_1 + \alpha x_2))}_{\bigcirc} \leq \underbrace{f((1-\alpha)g(x_1) + \alpha g(x_2))}_{\bigcirc}$

$\bigcirc \leq \bigcirc$

f convex, g convex, f non-decreasing $\Rightarrow f \circ g$ convex

f convex, g concave, f non-increasing $\Rightarrow f \circ g$ convex