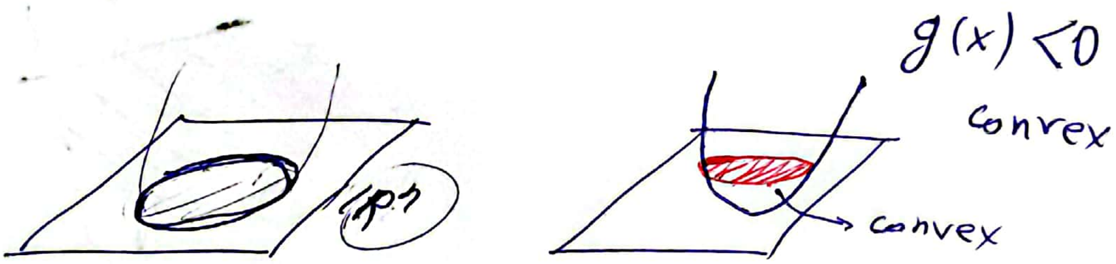
$$\min_x f(x) \quad \text{s.t.} \quad g_i(x) \leq 0 \quad i=1..p$$
$$h_i(x) = 0 \quad i=1..q$$

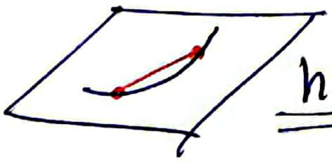
$$\min_x f(x) \quad \text{s.t.} \quad x \in C$$

Convex Optimization

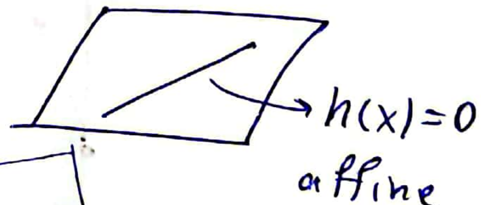
$$\boxed{\begin{array}{l} \min_x f(x) \quad \text{s.t.} \quad x \in C \\ f \text{ is a convex function} \\ C \text{ is convex set} \end{array}}$$



for C conex set, find ~~g(x)~~ a convex & differentiable $g: \mathbb{R}^n \to \mathbb{R}$ such that $\{x \mid g(x) \leq 0\} = C$.



$h(x) = 0$



$$\min_x f(x) \quad \text{s.t.} \quad g_i(x) \leq 0 \quad i=1-p$$
$$h_i(x) \leq 0 \quad i=1-q$$

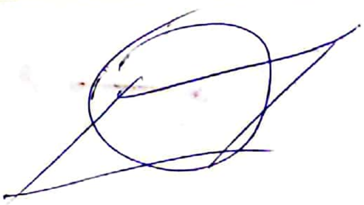
$\Rightarrow h(x)$: affine

$h(x) = a^T x + b$

f: convex

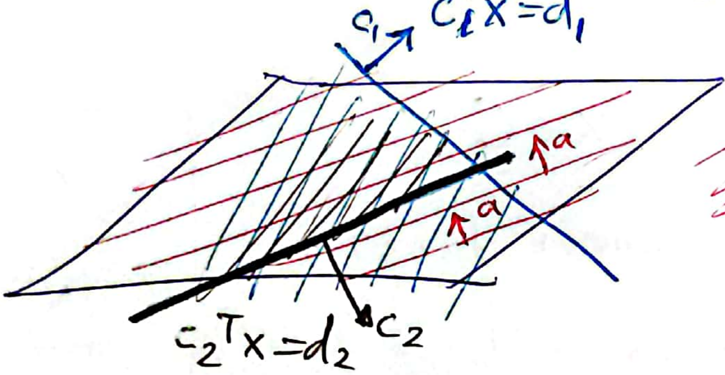
g_i : convex

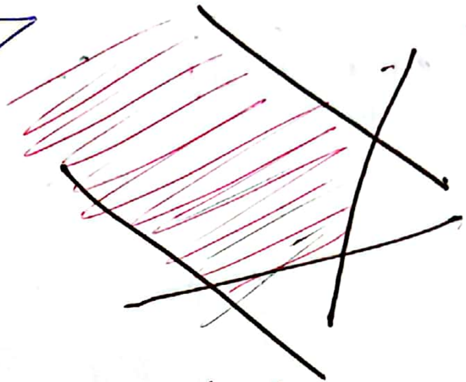
h_i: affine $\quad h_i(x) = a_i^T x + b_i$

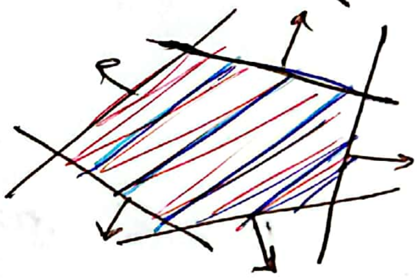


Linear programming

$$\min_{x} \ \vec{a}^T x \cancel{+ b} \qquad \text{s.t.} \quad \boxed{\begin{array}{l} \vec{c}_i^T x \leq d_i \\ \vec{v}_i^T x = u_i \end{array}}$$







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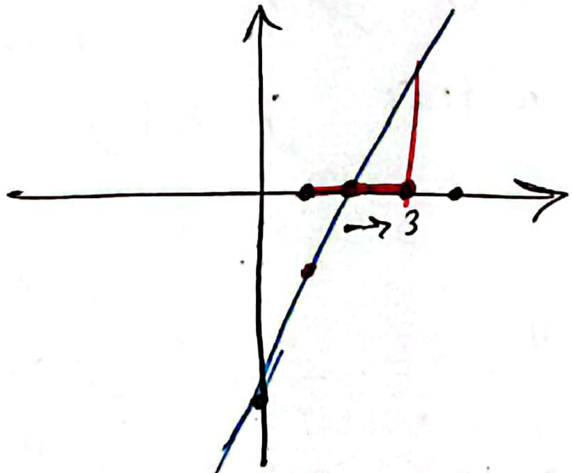
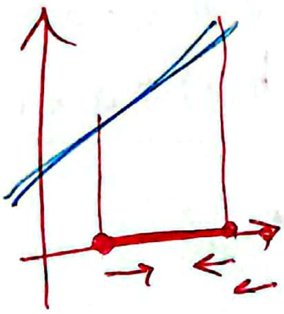
min $2x - 4$ subject to

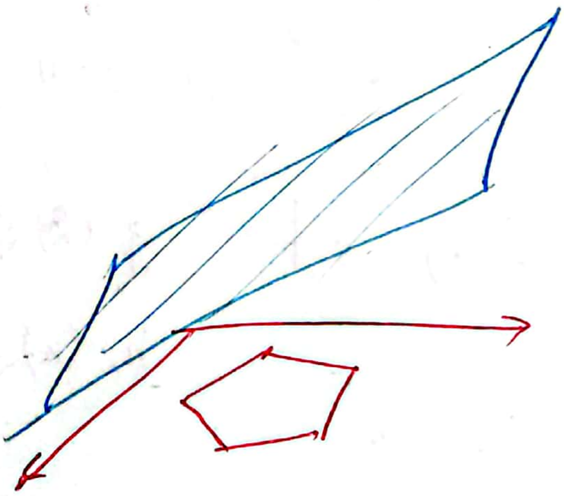
$x \geq 1$

$x \leq 3$

$x \leq 4$







$\min \ \vec{a}^T x \qquad \vec{c}_i^T x \leq d_i \quad i = 1 \cdots p$ MA27 III

$\vec{h}_i^T x = g_i \quad i = 1 \cdots q$

$$\begin{bmatrix} c_1^T \\ c_2^T \\ \vdots \\ c_p^T \end{bmatrix} x \leq \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_p \end{bmatrix} \qquad \cancel{Hx \leq g}$$

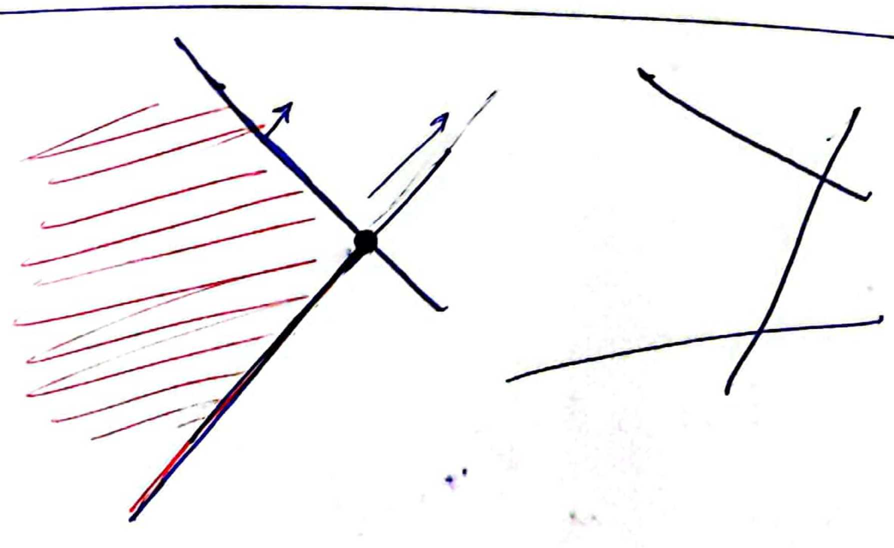
$$C x \leq \vec{d} \qquad Hx = \vec{g}$$

$$h_i^T x = g_i \equiv \begin{cases} h_i^T x \geq g_i \\ h_i^T x \leq g_i \end{cases} \equiv \begin{cases} -h_i^T x \leq -g_i \\ h_i^T x \leq g_i \end{cases}$$

$$\min \ \vec{a}^T x \quad \text{s.t.} \quad \cancel{c_i^T x \leq} \quad Cx \leq d$$

$$\min \ \vec{a}^T x \quad \text{s.t.} \quad Cx \leq d, \quad x \geq 0$$

$$\max_x \ \vec{a}^T x \quad \text{s.t} \quad Cx \leq d$$



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Quadratic Programming

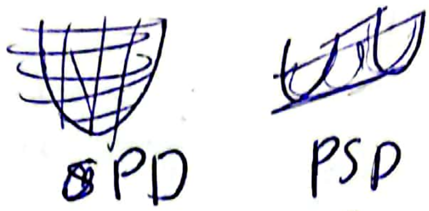
MA 27 (IV)

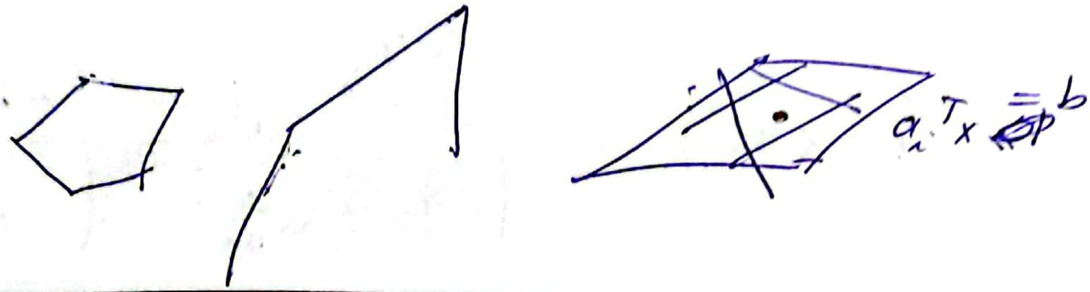
$$\min_x \ \frac{1}{2} x^T H x + g^T x \quad \text{s.t.} \quad \cancel{a_1^T x}$$

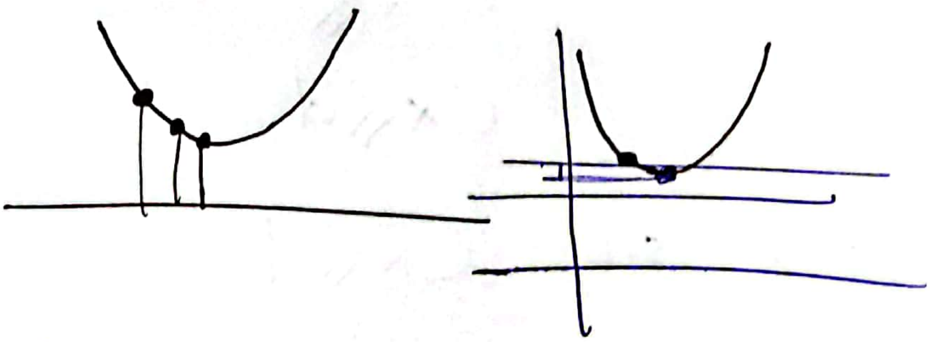
$$A x \leq \vec{b} \quad \begin{cases} a_1^T x \leq b_1 \\ a_2^T x \leq b_2 \end{cases}$$

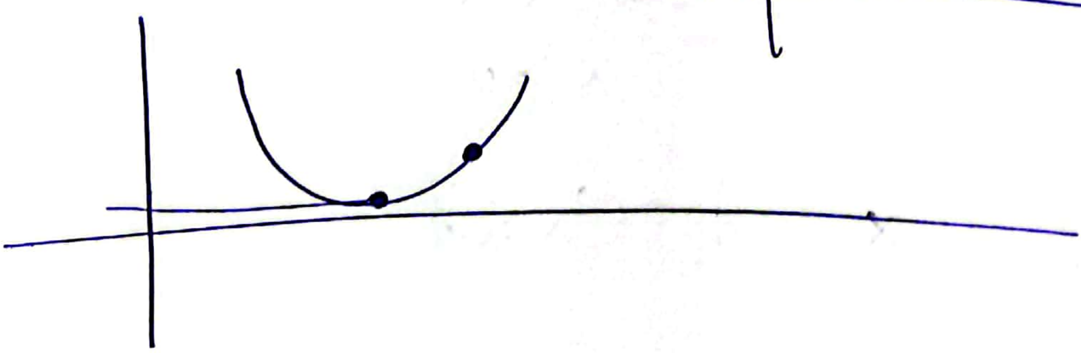
$$C \vec{x} = \vec{d} \quad \begin{cases} c_1^T x = d_1 \\ c_2^T x = d_2 \end{cases}$$

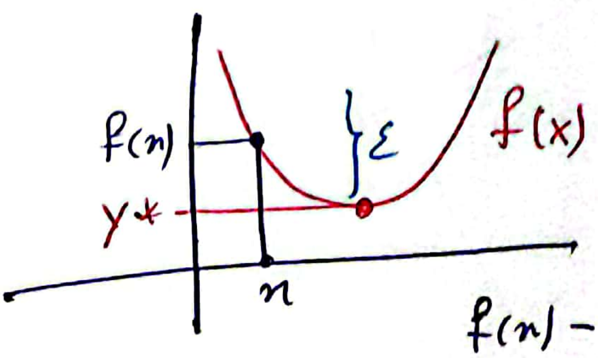
H positive definite





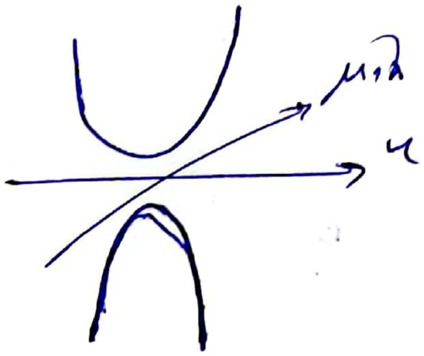


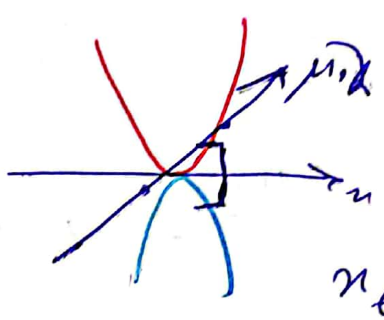




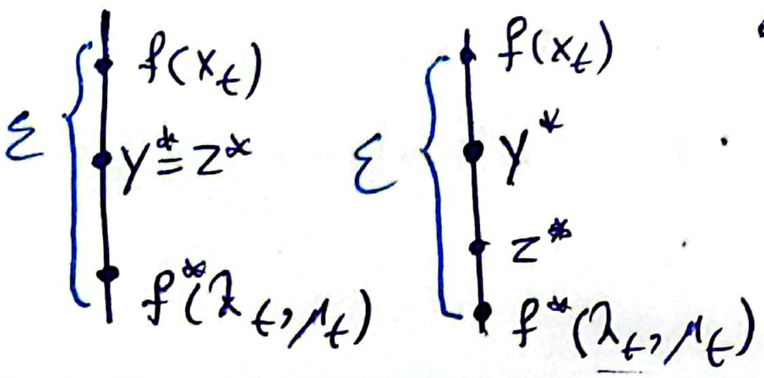
$f(n) - y^* < \varepsilon$

n is $\underline{\varepsilon\text{-suboptimal}}$





x_t, λ_t, μ_t



$\underline{f(x_t) - f^*(\lambda_t, \mu_t) \leq \varepsilon}$

Dual of a QP

$$\min_x \tfrac{1}{2} x^T H x + v^T x \quad \text{s.t.} \quad Ax \leq b$$

$a_1^T x \leq b_1$
$\vdots$
$a_m^T x \leq b_m$

$x \in \mathbb{R}^n \quad H \in \mathbb{R}^{n \times n} \quad v \in \mathbb{R}^n \qquad A \in \mathbb{R}^{m \times n} \quad b \in \mathbb{R}^m$

n variables, m constraints

$$\mathcal{L}(x,\lambda) = \tfrac{1}{2} x^T H x + v^T x + \lambda^T (Ax - b)$$
$$= \tfrac{1}{2} x^T H x + (v^T + \lambda^T A) x + \lambda^T b$$
$$= \tfrac{1}{2} x^T H x + \underbrace{(v + A^T \lambda)^T}_{\vec{c}} x + \lambda^T b$$

$$L(x,\lambda) = \tfrac{1}{2} x^T H x + c^T x + \lambda^T b \qquad f^*(\lambda) = \min_x L(x,\lambda)$$

$$\nabla_x L = \frac{\partial L}{\partial x} = \quad Hx + c = 0 \Rightarrow \boxed{x = -H^{-1} c}$$

symmetric $\quad$ H is PD $\Rightarrow H^{-1}$ exists, x is minimum

$$f^*(\lambda) = \tfrac{1}{2} (-H^{-1} c)^T H (-H^{-1} c) + c^T (-H^{-1} c) + \lambda^T b$$
$$= \tfrac{1}{2} c^T \underbrace{H^{-1} H}_{I} H^{-1} c - c^T H^{-1} c + \lambda^T b = -\tfrac{1}{2} c^T H^{-1} c + \lambda^T b$$
$$= -\tfrac{1}{2} (v + A^T \lambda)^T H^{-1} (v + A^T \lambda) + \lambda^T b$$

$$-\frac{1}{2}(v+A^T\lambda)^T H^{-1}(v+A^T\lambda)+\lambda^T b$$

$$-\frac{1}{2}\lambda^T \underbrace{A H^{-1} A^T}\lambda + (v^T A^T + b^T)\lambda - \frac{1}{2} v^T H^{-1} v$$

$$\boxed{\text{Dual } -\frac{1}{2}\lambda^T A H^{-1} A \lambda + (Av+b)^T\lambda \quad \text{s.t. } \lambda \geq 0}$$

m variable $\lambda_1 - \lambda_m$ m constr

Linear Programming

$$\min_x c^T x \quad \text{s.t. } Ax \leq b \quad Gx = d$$

$x \in \mathbb{R}^n$ $m \times n$

$$L(x,\lambda,\mu) = c^T x + \lambda^T(Ax-b) + \mu^T(Gx-d)$$

$$= (c + A^T\lambda + G^T\mu)^T x - \lambda^T b - \mu^T d$$

$$= \vec{q}^T x - p$$

$$\boxed{f^*(\lambda,\mu) = \min_x L(x,\lambda,\mu)}$$

$$= \begin{cases} -\lambda^T b - \mu^T d & c + A^T\lambda + G^T\mu = 0 \\ -\infty & c + A^T\lambda + G^T\mu \neq 0 \end{cases}$$

$$\max_{\lambda,\mu} -b^T\lambda - d^T\mu \quad \text{s.t. } \lambda \geq 0$$

$$A^T\lambda + G^T\mu = -c$$

$$\begin{bmatrix} -b \\ -d \end{bmatrix}^T \begin{bmatrix} \lambda \\ \mu \end{bmatrix} \qquad \begin{bmatrix} A^T & G^T \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = -\vec{c}$$

(VII)

$$x^* = \arg\min_x f(x) \quad \text{s.t} \quad g_i(x) \leq 0 \quad i=1-p$$
$$h_i(x) = 0 \quad i=1-q$$

$$\boxed{\lambda^*, \mu^*} = \arg\max_{\lambda,\mu} f^*(\lambda,\mu) \quad \text{s.t.} \quad \lambda_i \geq 0 \quad i=1 \dots p$$

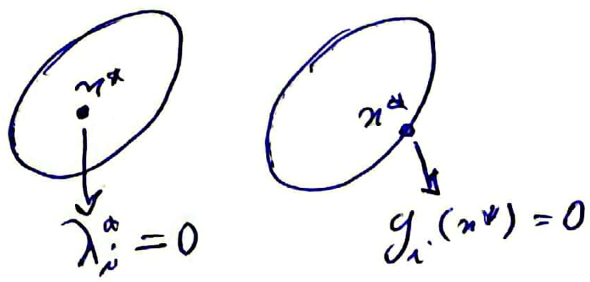
$$\text{duality gap}=0 \Rightarrow f^*(\lambda^*,\mu^*) = f(x^*)$$

$$f(x^*) = f^*(\lambda^*,\mu^*) = \min_x L(x,\lambda^*,\mu^*)$$
$$= \min_x f(x) + \sum_{i=1}^{p} \lambda_i^* g_i(x) + \sum_{i=1}^{q} \mu_i^* h_i(x)$$
$$\leq f(x^*) + \sum_{i=1}^{p} \underbrace{\lambda_i^*}_{\geq 0} \underbrace{g_i(x^*)}_{\leq 0} + \sum_{i=1}^{q} \mu_i^* \underbrace{h_i(x^*)}_{0}$$
$$= f(x^*) + \underbrace{\sum_{i=1}^{p} \lambda_i^* g_i(x^*)}_{\geq 0} \leq f(x^*)$$

$$\boxed{x^* \in \arg\min_x L(x,\lambda^*,\mu^*)}$$

$$f(x^*) = f(x^*) + \sum_{i=1}^{p} \lambda_i^* g_i(x^*)$$

$$\Rightarrow \underbrace{\sum_{i=1}^{p} \underbrace{\lambda_i^*}_{\geq 0} \underbrace{g_i(x^*)}_{\leq 0}}_{\leq 0} = 0 \Rightarrow \boxed{\lambda_i^* g_i(x^*) = 0}$$
$$i=1 \dots p$$



$x^* \in \arg\min L(x, \lambda^*, \mu^*) = f(x) + \sum \lambda_i g_i(x) + \sum \mu_i h_i(x)$ VIII

$f(x), g_i(x), h_i(x)$ differentiable $\Rightarrow$

$$\begin{cases} \nabla_x L(x^*, \lambda^*, \mu^*)\Big|_{x=x^*} = \nabla_x f(x^*) + \sum \lambda_i^* \nabla g_i(x^*) + \sum \mu_i^* \nabla h_i(x^*) = 0 \\ \lambda_i^* \nabla g_i(x^*) = 0 \\ \lambda^* \geqslant 0 \\ g_i(x^*) \leqslant 0 \\ h_i(x^*) = 0 \end{cases}$$

KKT conditions

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