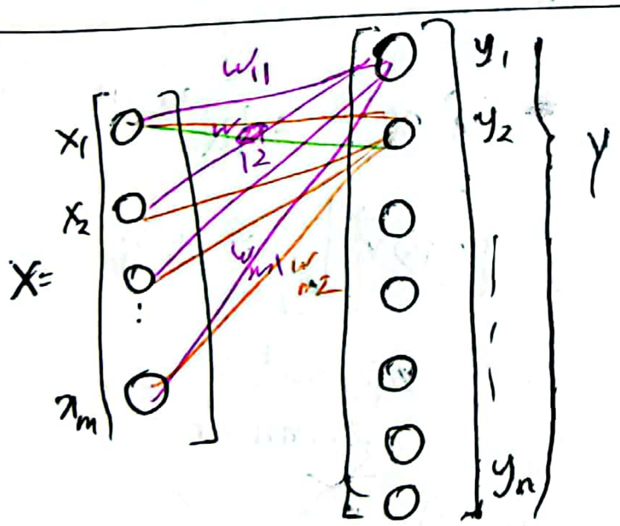
$f: \mathbb{R}^m \to \mathbb{R}^n$ linear

$f(\vec{x}) = A\vec{x} \qquad A \in \mathbb{R}^{n \times m}$

$$\underset{m \times n}{A} \; \underset{n \times p}{B} = \underset{m \times p}{C}$$

$$\underset{A}{\begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix}} \underset{B}{\begin{bmatrix} b_1 & b_2 & \cdots & b_p \end{bmatrix}} = \underset{C}{\begin{bmatrix} & & \\ & c_{ij} & \\ & & \end{bmatrix}}$$

$$c_{ij} = \langle a_i, b_j \rangle = a_i^T b_j$$



$$W = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1m} \\ w_{21} & w_{22} & \cdots & w_{2m} \\ & \vdots & & \\ w_{n1} & w_{n2} & & w_{nm} \end{bmatrix}$$

$$y_1 = WX$$

$$X = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_b \end{bmatrix} \longrightarrow \overset{Y}{\begin{bmatrix} y_1 & y_2 & \cdots & y_b \end{bmatrix}} = \begin{bmatrix} Wx_1, & Wx_2, \ldots, & Wx_b \end{bmatrix}$$

$$= Y = W \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_b \end{bmatrix} = WX$$

MAS (II)

$$AB = A\left[\vec{b}_1\ \vec{b}_2 \cdots \vec{b}_p\right] = \left[A\vec{b}_1, A\vec{b}_2, \dots, A\vec{b}_p\right]$$

$\left[\ \cancel{a_1^T}\right.$

$$AB = \underset{m\times n}{\begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix}} \underset{n\times p}{B} = \begin{bmatrix} a_1^T B \\ a_2^T B \\ \vdots \\ a_m^T B \end{bmatrix}$$

$$\underset{n\times m}{W} \quad \left[\vec{x}_1\ \vec{x}_2 \cdots \vec{x}_b\right]$$

$$\begin{bmatrix} \vec{x}_1^T \\ \vec{x}_2^T \\ \vdots \\ \vec{x}_b^T \end{bmatrix} W_{m\times n} = \begin{bmatrix} y_1^T \\ y_2^T \\ \vdots \\ y_n^T \end{bmatrix}$$

$\vec{u} \in \mathbb{R}^m$ $\quad$ $\cancel{b}\ \vec{v} \in \mathbb{R}^n$ $\quad$ $\vec{u} \otimes \vec{v} \in \mathbb{R}^{m\times n}$

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \qquad \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad \left[\vec{u} \otimes \vec{v}\right]_{ij} = u_i v_j$$

np.outer
np.inner

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \otimes \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 v_1 & u_1 v_2 \cdots & u_1 v_n \\ u_2 v_1 & u_1 v_2 & u_1 v_n \\ \vdots & & \ddots \\ u_m v_1 & u_m v_2 & u_m v_n \end{bmatrix} = \begin{bmatrix} v_1 \vec{u} & v_2 \vec{u} & \cdots & v_n \vec{u} \end{bmatrix}$$

$$\begin{bmatrix} u_1 \vec{v}^T \\ u_2 \vec{v}^T \\ \vdots \\ u_m \vec{v}^T \end{bmatrix} = \underset{m\times 1}{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}} \underset{1\times n}{\left[v_1\ v_2 \cdots v_n\right]} = \underset{m\times 1}{u}\ \underset{1\times n}{v^T}$$

MAS III

$$\underset{A}{\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}} \underset{B}{\begin{bmatrix} x & y \\ z & t \end{bmatrix}} = \begin{bmatrix} ax+bz & ay+bt \\ cx+dz & cy+dt \\ ex+fz & ey+ft \end{bmatrix}$$

$$= \begin{bmatrix} ax & ay \\ cx & cy \\ ex & ey \end{bmatrix} + \begin{bmatrix} bz & bt \\ dz & dt \\ fz & ft \end{bmatrix}$$

$$= \begin{bmatrix} a \\ c \\ e \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix} + \begin{bmatrix} b \\ d \\ f \end{bmatrix} \begin{bmatrix} z & t \end{bmatrix}$$

$$= \begin{bmatrix} a \\ c \\ e \end{bmatrix} \otimes \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b \\ d \\ f \end{bmatrix} \otimes \begin{bmatrix} z \\ t \end{bmatrix}$$

$$C = \underset{m\times n}{A}\ \underset{n\times p}{B} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} \vec{b}_1^T \\ \vec{b}_2^T \\ \vdots \\ \vec{b}_n^T \end{bmatrix}$$

$\vec{a}_i \in \mathbb{R}^m$

$\vec{b}_i \in \mathbb{R}^p$

$$\underset{m\times p}{C} = \underbrace{\underbrace{\vec{a}_1}_{m\times 1}\ \underbrace{\vec{b}_1^T}_{1\times p}}_{m\times p} + \underbrace{\vec{a}_2\ \vec{b}_2^T}_{m\times p} + \cdots + \underbrace{\vec{a}_n\ \vec{b}_n^T}_{m\times p}$$

$$= \sum_{i=1}^{n} \vec{a}_i\, \vec{b}_i^T = \sum_{i=1}^{n} \vec{a}_i \otimes \vec{b}_i$$

$$\underset{m\times n}{A} = \begin{bmatrix} \underset{m\times n_1}{A_1} & \underset{m\times n_2}{A_2} \end{bmatrix}$$

$n_1 + n_2 = n$

$$B_{n\times p} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \begin{matrix} \}\, n_1 \\ \}\, n_2 \end{matrix} \qquad B_1: n_1\times p, \quad B_2: n_2\times p$$

$$AB = \begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

$$= \begin{bmatrix} A_1B_1 + A_2B_2 \end{bmatrix}$$

Scanned with CamScanner

MA 5 IV

$$\begin{array}{c} m_1 \\ m_2 \end{array}\underbrace{\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}}_{n} \begin{bmatrix} \overbrace{B_1}^{p_1} & \overbrace{B_2}^{p_2} \end{bmatrix} \} n = \begin{bmatrix} A_1 B_1 & A_1 B_2 \\ A_2 B_1 & A_2 B_2 \end{bmatrix}.$$

$$\begin{array}{c} m_1 \\ m_2 \end{array}\begin{bmatrix} \overbrace{A_{11}}^{n_1} & \overbrace{A_{12}}^{m_2} \\ A_{21} & A_{22} \end{bmatrix} \begin{array}{c} m_1 \\ n_2 \end{array}\begin{bmatrix} B_{11} & B_{12} & B_{13} \\ \hline B_{21} & B_{22} & B_{23} \end{bmatrix}$$

$$= \left[\begin{array}{c|c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} & A_{11}B_{13} + A_{12}B_{23} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} & A_{21}B_{13} + A_{22}B_{23} \end{array}\right]$$

$$\begin{bmatrix} 1 & -2 \\ 2 & -4 \\ 3 & -6 \\ 4 & -8 \end{bmatrix}$$

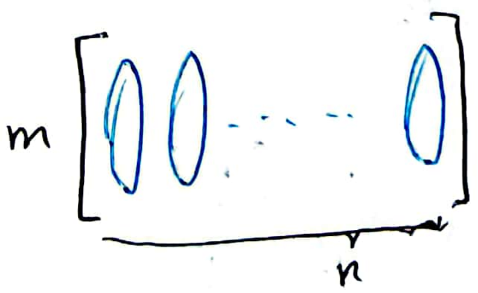
column rank = 1

row rank = 1

$$\begin{bmatrix} 1 & 0 & 4 \\ 2 & 0 & 5 \\ 3 & 0 & 6 \end{bmatrix}$$

column rank = 2

row rank = 2



column rank $\leq m$

column rank $\leq n$

colum rank $\leq \min(m,n)$

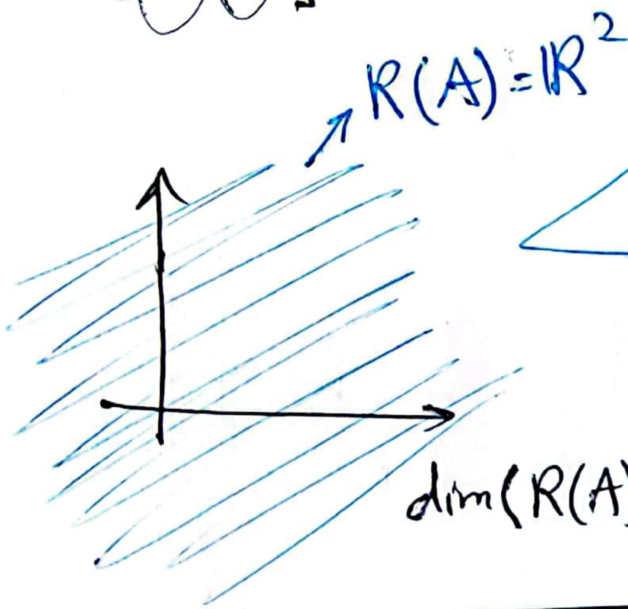
row-rank $\leq \min(m,n)$

MAS (V)

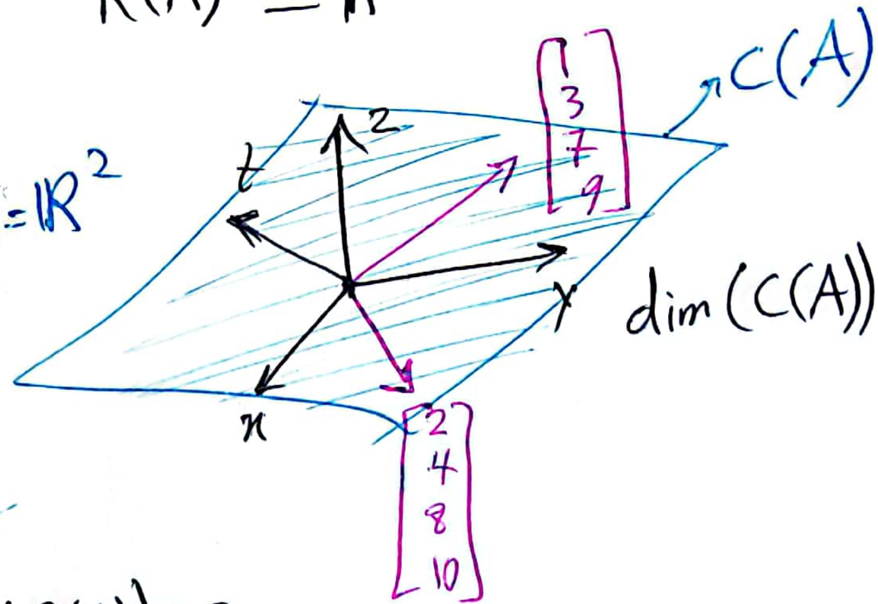
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 7 & 8 \\ 9 & 10 \end{bmatrix}$$

$C(A) \subseteq \mathbb{R}^4$

$R(A) \subseteq \mathbb{R}^2$



$\dim(R(A)) = 2$



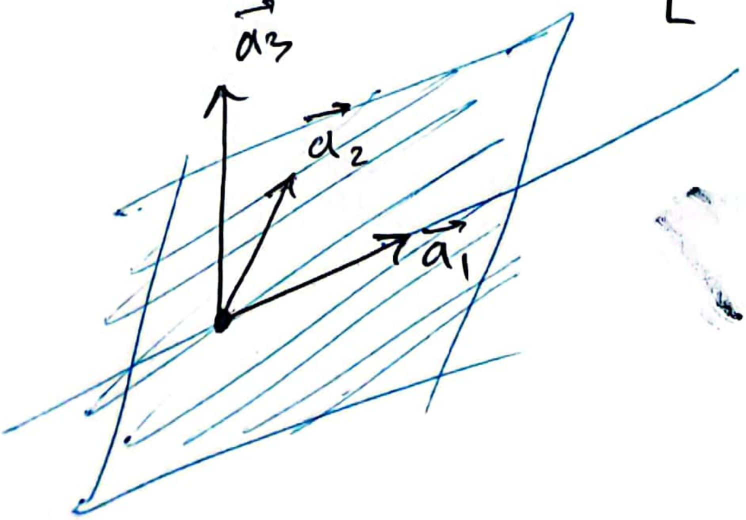
$\dim(C(A)) = 2$

$A \in \mathbb{R}^{m \times n}$ $\qquad$ $\text{rank}(A) \leq \min(m,n)$

$\text{rank}(A) = \min(m,n)$ full-rank

$\text{rank}(A) < \min(m,n)$ rank deficient

$A \in \mathbb{R}^{3\times 3}$ $\qquad$ $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix}$ $\qquad$ $\vec{a}_1, \vec{a}_2, \vec{a}_3 \in \mathbb{R}^3$



$\text{prob}(\vec{a}_2 \in \text{span}(\vec{a}_1)) = 0$

$\text{prob}(\vec{a}_3 \in \text{span}(\vec{a}_1, \vec{a}_2)) = 0$