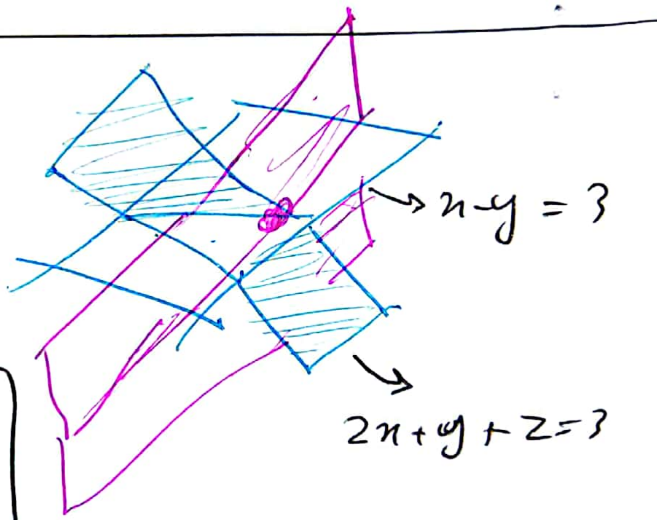
MA6 (I)

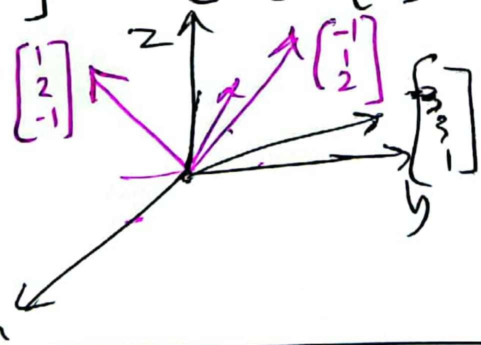
$$A = \begin{bmatrix} \; \end{bmatrix} \in \mathbb{R}^{m \times n}$$

full-rank	$\text{rank}(A) = \min(m,n)$
rank-deficient	$\text{rank}(A) < \min(m,n)$
full column rank	$\text{rank}(A) = n$
full row rank	$\text{rank}(A) = m$

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$$



$$x \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + z \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$$



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if $A \in \mathbb{R}^{n \times n}$ (square) has full rank $(\text{rank}(A) = n)$

we say that A is non-singular (invertible).

$\text{rank}(A) < n \Rightarrow A$ singular.

MA6 (II)

$A \in \mathbb{R}^{n\times n}$, non-singular $\left(\text{rank}(A)=n\right)$

what are the solutions to $\underbrace{A}_{\in \mathbb{R}^{n\times n}} \underbrace{x}_{\in\mathbb{R}^n} = \underbrace{b}_{\in\mathbb{R}^n}$

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^n \qquad f(\vec{x}) = \underbrace{A}_{n\times n}\vec{x} \qquad A$ is non-singular

one-to-one

$$f(\vec{x}_1) = f(\vec{x}_2) \Rightarrow A\vec{x}_1 = A\vec{x}_2 \Rightarrow A(\underbrace{\vec{x}_1 - \vec{x}_2}_{\vec{d}}) = 0.$$

$$A\vec{d} = 0 \Rightarrow \left[\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n\right]\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n\end{bmatrix} = 0$$

$$\Rightarrow d_1\vec{a}_1 + d_2\vec{a}_2 + \cdots + d_n\vec{a}_n = 0 \Rightarrow d_1 = d_2 = \cdots = d_n = 0$$

$\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are linearly independent

$$\Rightarrow \vec{d} = \vec{0} \Rightarrow \vec{x}_1 - \vec{x}_2 = 0 \Rightarrow \vec{x}_1 = \vec{x}_2$$

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onto: Let $\vec{y}$ be any vector in $\mathbb{R}^n$.

$\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are n linearly independent vectors in $\mathbb{R}^n$, and thus, they span $\mathbb{R}^n$. There exist scalars $x_1, x_2, \ldots, x_n \in \mathbb{R}$ such that $x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{y}$

$$\left[\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n\right]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n\end{bmatrix} = \vec{y} \Rightarrow A\vec{x} = \vec{y} = f(\vec{x})$$

$\Rightarrow f$ is onto. $\square$

Scanned with CamScanner

$f(x) = Ax$ $\quad$ A : square & non-singular

$\Rightarrow f$ is
- one-to-one $\Rightarrow$ $\boxed{Ax=b \text{ has at most one solution}}$
- onto $\Rightarrow$ $\boxed{f(x)=b \quad Ax=b \text{ has at least one solution}}$

one-to-one & onto $\Rightarrow Ax=b$ has exactly one solution

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f: one-to-one & onto $\Rightarrow f$ is inertible

$\exists f^{-1}$: $\boxed{f^{-1}(f(x)) = x \quad \forall x \in \mathbb{R}^n}$ $\boxed{f(f^{-1}(y)) = y \quad \forall y \in \mathbb{R}^n}$

$f: \mathbb{R}^n \to \mathbb{R}^n \qquad f(x) = Ax$

Is f^{-1} linear?

$$f^{-1}(a\vec{y}_1 + b\vec{y}_2) = af^{-1}(y_1) + bf^{-1}(y_2) \quad =?$$

f is onto $\Rightarrow \exists \vec{x}_1, \vec{x}_2 \in \mathbb{R}^n \quad \vec{y}_1 = f(\vec{x}_1), \ \vec{y}_2 = f(\vec{x}_2)$

$$f^{-1}(a\vec{y}_1 + b\vec{y}_2) = f^{-1}(af(\vec{x}_1) + bf(x_2))$$

$$\underset{f \text{ linear}}{=} f^{-1}(f(ax_1 + bx_2)) = a\vec{x}_1 + b\vec{x}_2$$

$$= af^{-1}(\vec{y}_1) + bf^{-1}(y_2)$$

$\Rightarrow f^{-1}$ is also linear.

$f^{-1}: \mathbb{R}^n \to \mathbb{R}^n$ is linear $\Rightarrow \exists B \in \mathbb{R}^{n\times n} \Rightarrow f^{-1}(\vec{y}) = B\vec{y}$

B is called the inverse of A.

MA6 IV

$f: \mathbb{R}^n \to \mathbb{R}^n \quad f(\vec{x}) = A\vec{x}$ A non-singuller

$f^{-1}: \mathbb{R}^n \to \mathbb{R}^n \quad f^{-1}(\vec{y}) = \cancel{A}B\vec{y}$

$$f(f^{-1}(y)) = y \Rightarrow AB\vec{y} = \vec{y} \quad \forall \vec{y} \in \mathbb{R}^n$$

$$f^{-1}(f(x)) = x \Rightarrow BA\vec{x} = \vec{x} \quad \forall \vec{x} \in \mathbb{R}^n$$

$$AB \underset{\vec{e}_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}} = \underset{\vec{e}_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}} \qquad AB \underset{\vec{e}_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}} = \underset{\vec{e}_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}} \quad \cdots \quad AB \underset{\vec{e}_n}{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}} = \underset{\vec{e}_n}{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}$$

$$AB\left[\vec{e}_1\ \vec{e}_2 \cdots \vec{e}_n\right] = \left[\vec{e}_1\ \vec{e}_2 \cdots \vec{e}_n\right]$$

$$AB \begin{bmatrix} 1 & & \emptyset \\ & \ddots & \\ \emptyset & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \emptyset \\ & \ddots & \\ \emptyset & & 1 \end{bmatrix} \Rightarrow ABI = I$$

$$\Rightarrow AB = I$$

$BA = I$

B is denoted by A^{-1} and is called the inverse of A.

$A \in \mathbb{R}^{n\times n}$ non-singular $\Rightarrow \exists A^{-1} \in \mathbb{R}^{n \times n}$

$$A^{-1}A = AA^{-1} = I$$

How to solve ~~for~~ $A\vec{x} = \vec{b}$ A non-singular?

$\to$ find the inverse matrix A^{-1}

$\to A^{-1}A\cancel{b}^{x} = A^{-1}b \Rightarrow \boxed{x = A^{-1}b}$

How to find A^{-1}? MA6 (V)

numpy.linalg.inv(A)

X = np.linalg.inv(A) @ b Bad way

X = np.linalg.solve(A, b) Good way

Eigen

What if A is singular? $A \in \mathbb{R}^{n\times n}$

$\text{rank}(A) < n$

$$A = [\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n] \Rightarrow \vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n \text{ are dependent}$$

$$\Rightarrow \text{span}(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n) \neq \mathbb{R}^n$$

$$C(A) \neq \mathbb{R}^n$$

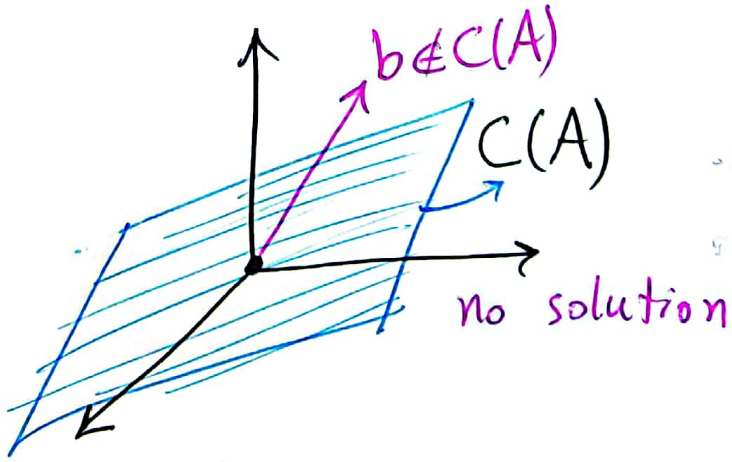
$$A\vec{x} = \vec{b}$$

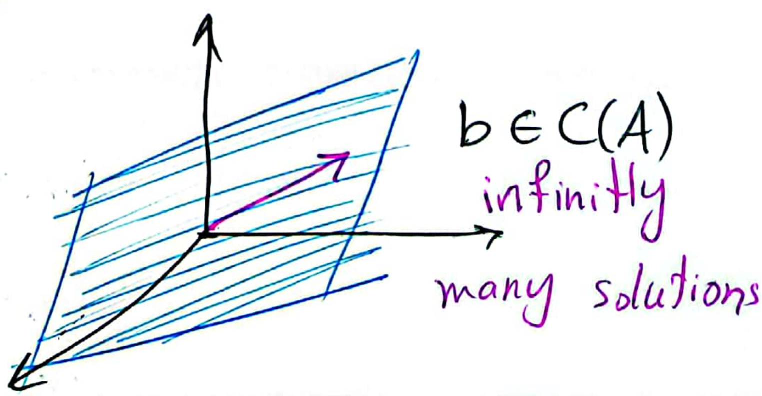
$\vec{x} \in \mathbb{R}^n$

$\vec{b} \in \mathbb{R}^n$

$$A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{b}$$

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}$$





$A\vec{x} = \vec{b}$, $\vec{b} \notin C(A) \Rightarrow$ no solution

MA6 (VI)

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & - & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{b}$$

$A \in \mathbb{R}^{m \times n}$

$\vec{a}_i \in \mathbb{R}^m$

$b \in \mathbb{R}^m$

$$\sum x_i \vec{a}_i = \vec{b}$$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$

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m equations, n unknowns $A \in \mathbb{R}^{m \times n}$

$$A\vec{x} = \vec{b} \quad \begin{cases} \vec{b} \notin C(A) \Rightarrow \text{No Solution!} \\ \vec{b} \in C(A) \Rightarrow \text{At least one solution} \end{cases}$$

$\vec{b} \in C(A) \Rightarrow \vec{b} \in \text{span}(\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n)$

$$\Rightarrow \vec{b} = \sum_{i=1}^{n} x_i \vec{a}_i$$

Let $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \Rightarrow \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & - & \vec{a}_n \end{bmatrix} \vec{x} = \vec{b}$

$$\Rightarrow A\vec{x} = \vec{b}$$

$$\boxed{\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{n} x_i \vec{a}_i}$$