scaling

$$f(\vec{x}) = f\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ a_n \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \circ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{a} \circ \vec{x} = \begin{bmatrix} a_1 x_1 \\ a_2 x_2 \\ \vdots \\ a_n x_n \end{bmatrix} = \mathrm{diag}(\vec{a})\, \vec{x}$$

element-wise product

Hadamard product

$$\vec{x} \circ \vec{y} = \mathrm{diag}(\vec{x})\, \vec{y} = \mathrm{diag}(\vec{y})\, \vec{x}$$

$$f(x) = A\vec{x} + \vec{a} \circ \vec{x} = \left(A + \mathrm{diag}(\vec{a})\right) \vec{x}$$

$$A\vec{x} = \vec{b}$$

$A \in \mathbb{R}^{m\times n}$, $\vec{x} \in \mathbb{R}^n$, $\vec{b} \in \mathbb{R}^m$

$A \in \mathbb{R}^{n\times n}$, nonsingular $\Longrightarrow$ $A\vec{x} = \vec{b}$ has a unique solution

---

$A \in \mathbb{R}^{n\times n}$ singular

$\vec{b} \notin C(A)$ no solutions

$\vec{b} \in C(A)$ at least one solution

A singular $\Longrightarrow$ $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}$ $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are linearly dependent

$\Longrightarrow$ $\exists\ x_1, x_2, \ldots, x_n \in \mathbb{R}$ $x_i \neq 0$ at least for one i

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{0}$$

$$\vec{x} \overset{\text{def}}{=} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \Longrightarrow \vec{x} \neq \vec{0},\ A\vec{x} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{0}$$

$\doteq$

A singular $\Longrightarrow$ $\exists\ \vec{x} \neq 0$ $A\vec{x} = 0$

$\vec{x}$ is a <u>null vector</u> of A

بردار تهی

$A\vec{x} = \vec{b}$

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$\doteq$

A singular $\Longrightarrow$ $\exists\ \vec{x} \neq 0$ $A\vec{x} = 0$

$\vec{x}$ is a null vector of A

بردار تهی

$A \in \mathbb{R}^{n\times n}$ singular

$$A\vec{x} = \vec{b}$$

$\vec{b} \in C(A) \Rightarrow$ at least one solution

$$\Rightarrow \exists\, \vec{x}_p \in \mathbb{R}^n \quad A\vec{x}_p = b$$

A singular $\Rightarrow \exists\, \vec{x}_n \in \mathbb{R}^n,\ \vec{x}_n \neq 0,\ A\vec{x}_n = \vec{0}$

$$x = \vec{x}_p + \alpha\, \vec{x}_n$$

$$A\vec{x} = A(\vec{x}_p + \alpha \vec{x}_n) = A\vec{x}_p + \alpha A\vec{x}_n = A\vec{x}_p = b$$

$\vec{x}_p + \alpha \vec{x}_n$ is a solution to $A\vec{x} = \vec{b}$ for all $\alpha \in \mathbb{R}$.

$A \in \mathbb{R}^{n\times n}$, singular

$$A\vec{x} = b \begin{cases} b \notin C(A) & \text{No solution} \\ b \in C(A) & \text{infinitly many solutions} \end{cases}$$

Matrix Decomposition

$$A = BC \qquad\qquad A = BCD$$

قطری
diagonal $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

upper-triangular ← بالا مثلثی

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

Lower triangular $\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$

پایین مثلثی

LU Decomposition

$A = L U$

L: Lower tri

U: Upper tri

$$\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

Lower-tri

$a x_1 = 5 \Rightarrow x_1 = \frac{5}{a}$ ✓

$b x_1 + c x_2 = 6$

$\Rightarrow x_2 = \frac{6 - b x_1}{c}$ ✓

$x_3 =$ ✓

Backward Substitution

$Ax = b$ $\quad A$ triangular $\Longrightarrow$ EASY!

$Ax = b \qquad A = LU$

$L \underbrace{U x}_{y} = b$

$Ly = b \Rightarrow y =$ ✓

$Ux = y \Rightarrow x =$ ✓

np.linalg.solve

Permutation Matrix

ماتریس جایگشت

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ c \\ a \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix} = \begin{bmatrix} e & f & g & h \\ i & j & k & l \\ a & b & c & d \end{bmatrix}$$ MA 7 IV

PA permutes the rows of A

P: permutation matrix

AP permutes the columns of A

$P^T = P^{-1} \Rightarrow$ Permutation matrix is orthogonal

متعامد

$A^{-1} = A^T$

$AA^T = A^TA = I$

LU Decomposition

For Any matrix $A \in \mathbb{R}^{n \times n}$ ~~there exists~~ can be written as

$$PA = LU$$

P: permutation, L: Lower-tri, U: Upper-tri

$$A = \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 6 & 2 & 0 \\ 12 & 4 & 4 \end{bmatrix}}_{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}} \begin{bmatrix} 4 & 8 & 20 \\ 0 & 5 & 25 \\ 0 & 0 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix}\begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix} = \begin{bmatrix} s_1\vec{a}_1 & s_2\vec{a}_2 & s_3\vec{a}_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 12 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 28 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = L \quad D \quad U$$

$$PA = LDU$$

$U \to$ upper-tri

L: lowerr; D: diagonal

with unit diagonal elements

$A \in \mathbb{R}^{m \times n}$

~~column~~ $\cancel{\mathcal{N}(A)}$

column space	$C(A)$	range
row space	$R(A)$	
null space	$N(A)$	

$N(A) = \{\vec{x} \mid A\vec{x} = \vec{0}\}$ Is $N(A)$ linear?

$$\vec{x} \in N(A) \Rightarrow A\vec{x} = \vec{0} \Rightarrow A(\alpha\vec{x}) = \vec{0} \Rightarrow \alpha\vec{x} \in N(A)$$

$$\vec{x}, \vec{y} \in N(A) \Rightarrow \begin{cases} A\vec{x} = \vec{0} \\ A\vec{y} = \vec{0} \end{cases} \Rightarrow A\vec{x} + A\vec{y} = \vec{0} \Rightarrow A(\vec{x} + \vec{y}) = \vec{0}$$

$$\vec{x} + \vec{y} \in N(A)$$

$\boxed{N(A) \text{ is a linear subspace}}$

$$A\vec{x} = \vec{b}$$

$m \times n$ $\quad \in \mathbb{R}^n \quad \in \mathbb{R}^m$

$\vec{b} \notin C(A)$ No Solution

$\vec{b} \in C(A) \Rightarrow$ there exist a linear combination of the column of A creating $\vec{b}$

$\Rightarrow \exists \vec{x}_p \quad A\vec{x}_p = b$

$x_n \in N(A) \Longrightarrow A(\vec{x}_p + \vec{x}_n) = A\vec{x}_p + A\vec{x}_n = A\vec{x}_p = b$

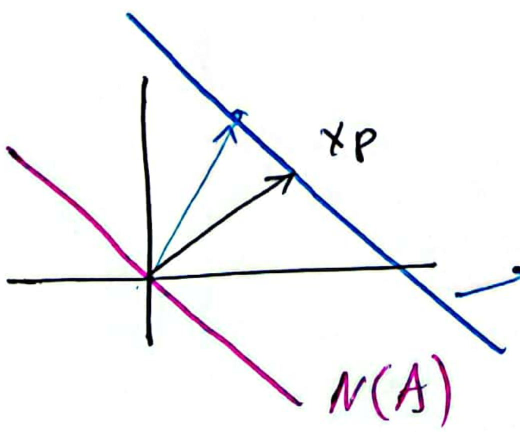
$\Longrightarrow$ $\vec{x}_p + \vec{x}_n$ is a solution to $A\vec{x} = \vec{b}$ for all $\vec{x}_n \in N(A)$

Let $\vec{x}$ is a solution to $A\vec{x} = \vec{b}$

$$\left.\begin{aligned} A\vec{x} &= b \\ A\vec{x}_p &= b \end{aligned}\right\} \Rightarrow A(\vec{x} - \vec{x}_p) = \vec{0}$$

$\vec{x} - \vec{x}_p \in N(A)$

$x = x_p + (\vec{x} - \vec{x}_p) = x_p + x_n \quad x_n \in N(A)$



EMA 7 VII

$A\vec{x} = \vec{b} \rightarrow \mathbb{R}^m$

A: $m \times n$, $\vec{x} \in \mathbb{R}^n$

$[A][\vec{x}] = [b] \Rightarrow m \geq n$

A has full column rank $(\text{rank}(A) = n)$

$\Rightarrow A = [\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n]$ columns of A are linearly independent

$\Rightarrow A\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}$

$\Rightarrow N(A) = \{\vec{0}\}$

$$\begin{cases} b \notin C(A) & \text{No solution} \\ b \in C(A) & \text{A unique solution} \end{cases}$$

$A \in \mathbb{R}^{m \times n}$ full row rank $(\text{rank}(A) = m)$

$[\ A\ ][\vec{x}] = [\vec{b}] \in \mathbb{R}^m$ $\quad m \leq n$

$\Rightarrow b \in C(A) \Rightarrow$ at least one solution

$\dim(N(A)) = n - \text{rank}(A)$

$A \in \mathbb{R}^{m \times n}$ $\left.\begin{array}{l}\text{full row rank} \\ \text{full column rank}\end{array}\right\} \Rightarrow m = n$

A square

A non-singular

full row rank $\Rightarrow$ at least one solution

full col rank $\Rightarrow$ no solution or a unique solution

$\Rightarrow$ A unique solution!

MA7 VIII

$A \in \mathbb{R}^{m \times n}$ rank-deficient

$r = \text{rank}(A) < \min(m, n)$

$Ax = b \Rightarrow b \notin C(A) \Rightarrow$ No Solution

$b \in C(A)$ $\quad$ A has dependent columns

$\boxed{\dim(N(A)) = n - \text{rank}(A) > 0} \Rightarrow \exists\, \vec{x}_n \neq 0$

$A\vec{x}_n = \vec{0}$

$N(A) \neq \{\vec{0}\}$

$N(A) \supset \{\vec{0}\}$ $\left\{ \begin{array}{l} \{\vec{0}\} \subseteq N(A) \\ \{\vec{0}\} \neq N(A) \end{array} \right.$

---

set of solutions to $Ax = b$, $\vec{b} \neq \vec{0}$

$S = \{x \mid Ax = b\}$ is linear subspace?

$\vec{x} \in S \Rightarrow A\vec{x} = \vec{b} \Rightarrow A(\alpha\vec{x}) = \alpha\vec{b} \neq \vec{b}$

$\alpha\vec{x} \notin S$

$\vec{x}_p$ is a solution to $A\vec{x} = \vec{b}$

$S' = S - \vec{x}_p = \{x - x_p \mid x \in S\}$ is linear