$2x+3y-z=0$ | $x=2$ | $x=4$ | $x=2\alpha$

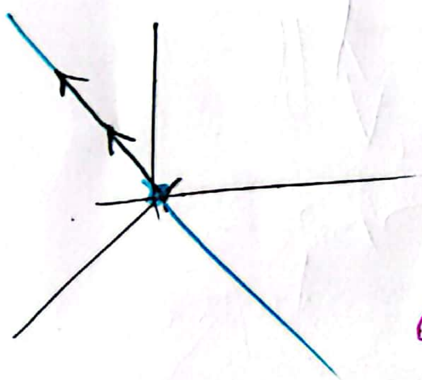
$x-2y-4z=0$ | $y=-1$ | $y=-2$ | $y=-\alpha$

| $z=1$ | $z=2$ | $z=\alpha$

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & -2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a solution, so is $\alpha \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for all $\alpha \in \mathbb{R}$.

$$A\vec{x} = \vec{0}$$ معادلات همگن



$A \in \mathbb{R}^{m\times n}$, $\vec{x} \in \mathbb{R}^n$, $\vec{0} \in \mathbb{R}^m$

Only the direction of $\vec{x}$ matters!

All Solutions to $A\vec{x}=\vec{0}$ $\equiv N(A)$

How to represent a linear subspace $S \subseteq \mathbb{R}^n$?

$\dim(S) = d$

1- A basis $V = \begin{bmatrix} v_1 & v_2 & \cdots & v_d \end{bmatrix}$ $\equiv VH$

$H \in \mathbb{R}^{d\times d}$, invertible

$V' = VH$ is also a basis.

2- (Implicitly) as the null space of a matrix.

$$S = N(A)$$

A has full-row-rank $\begin{array}{cc} & n \\ n-d & \begin{bmatrix} \; A \; \end{bmatrix} \end{array}$

$A \in \mathbb{R}^{d \times n}$ $\quad$ $\text{rank}(A) = \cancel{d}\; n-d$

3- Grassmann Coordinates

vector of determinants of $d \times d$ submatrices.

$$\begin{array}{cc} & d \\ n & \begin{bmatrix} | & | & | & | \\ | & | & | & | \\ | & | & | & | \end{bmatrix} \end{array} \qquad \binom{n}{d}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Dot product $x, y \in \mathbb{R}^n$ $\quad x^T y = \cancel{\;}\, y^T x \in \mathbb{R}$

inner product $\langle x, y \rangle$

Vector Space V + Inner Product $\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{R}$

Inner Product Space.

Lenght of a vector

$\vec{x} \in \mathbb{R}^n$ $\quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $\quad \|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

$= \sqrt{\vec{x}^T \vec{x}} = \sqrt{\langle \vec{x}, \vec{x} \rangle}$

General Vector Space (inner product space)

$$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$$

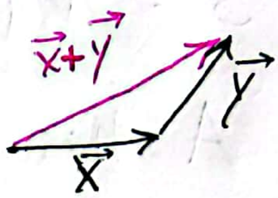
$\|\cdot\| : V \to \mathbb{R}$

$\|\vec{x}\| \geq 0$

$\|\vec{x}\| = 0 \iff \vec{x} = \vec{0}$

$\|\alpha \vec{x}\| = |\alpha| \, \|\vec{x}\|$

$\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$



$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $\quad \vec{x} = [x_1 \ x_2 \cdots x_n]^T$. $\quad \vec{x}^T = [x_1 \ x_2 \cdots x_n]$

l_2-norm $\quad \|\vec{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\vec{x}^T \vec{x}}$

Euclidean norm

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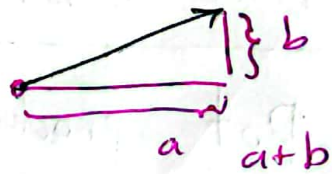
l_1-norm $\quad \|\vec{x}\|_1 = |x_1| + |x_2| + \cdots + |x_n|$



~~l^p-norm~~ l^p-norm $\quad \|\vec{x}\|_p = \sqrt[p]{|x_1|^p + |x_2|^p + \cdots + |x_n|^p}$

$\left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\right)^{1/p}$

$p \geq 1$

$\lim_{p \to \infty} \|\vec{x}\|_p = \max(|x_1|, |x_2|, \ldots, |x_n|) = \|\vec{x}\|_\infty$

Vector Space + Norm

$\Rightarrow$ Normed Vector Space

Inner product Space $\langle \vec{x}, \vec{y} \rangle$

$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$ $\Rightarrow$ Normed Vector space numpy.linalg.norm

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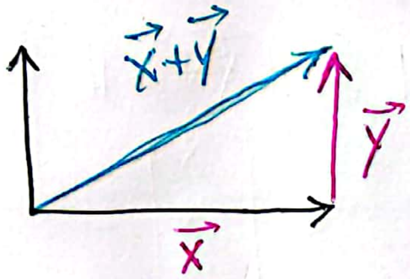
Inner Product → زاویه بین دو بردار

→ orthogonality تعامد



Check orthogonality using lenght

(قضیه فیثاغورث

Pythagorean theorem

$$x \perp y \Rightarrow \|x+y\|^2 = \|x\|^2 + \|y\|^2$$

$$\langle x+y, x+y \rangle = \langle x,x \rangle + \langle y,y \rangle$$

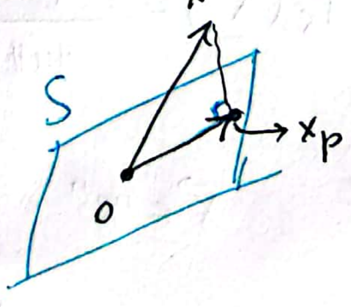
$$\langle x, x+y \rangle + \langle y, x+y \rangle = \circ + \circ$$

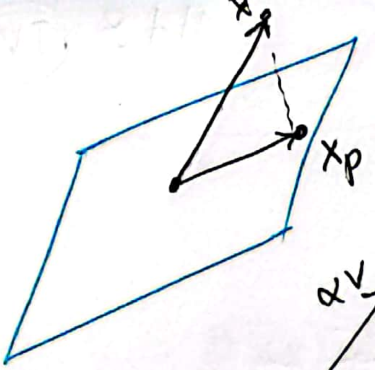
$$\cancel{\langle x,x \rangle} + \langle x,y \rangle + \langle y,x \rangle + \cancel{\langle y,y \rangle} = \cancel{\langle x,x \rangle} + \cancel{\langle y,y \rangle}$$

$$\boxed{\vec{x} \perp \vec{y} \qquad \langle \vec{x}, \vec{y} \rangle = 0}$$

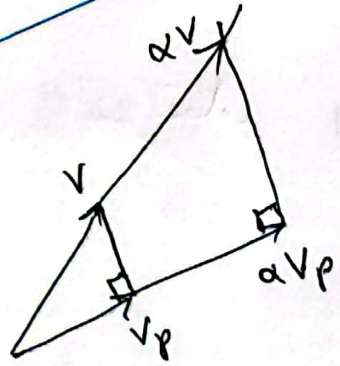
(orthogonal) projection

تصویر کردن ، افکنش

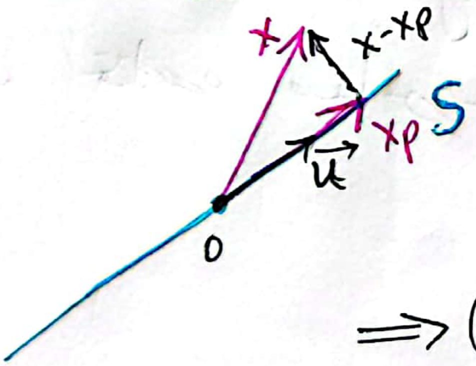




$x_p = P_S(x)$ $\quad$ P_S linear? YES!



1- S is a 1D linear subspae



$\vec{x}_p = \alpha \vec{u}$ $\qquad \alpha = ?$

$x - x_p \perp u \Rightarrow (x - x_p)^T u = 0$

$$\Rightarrow (\vec{x} - \alpha \vec{u})^T \vec{u} = 0 \Rightarrow \underline{x^T u} - \alpha\, \underline{u^T u} = 0$$

$$\boxed{\alpha = \frac{x^T u}{u^T u} = \frac{x^T u}{\|u\|^2}}$$

$$x_p = \alpha \vec{u} = \left(\frac{x^T u}{\|u\|^2}\right) \vec{u} = P_s(\vec{x})$$

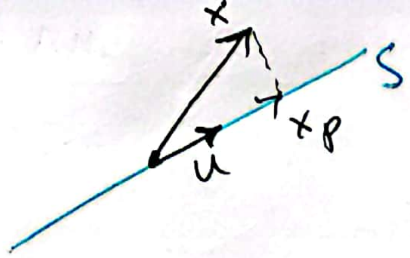
$$P_s(x) = \left(\frac{u^T x}{\|u\|^2}\right) \vec{u} = \frac{1}{\|u\|^2} (u^T x) \vec{u}$$

خطی $P_s(x) = Mx \qquad M = ?$

$$P_s(\vec{x}) = \frac{1}{\|u\|^2} \underbrace{(\underbrace{\vec{u}^T}_{1\times n} \underbrace{\vec{x}}_{n\times 1})}_{1\times 1} \underbrace{\vec{u}}_{n\times 1} = \frac{1}{\|u\|^2} \underbrace{\vec{u}}_{n\times 1} \underbrace{(\underbrace{\vec{u}^T}_{1\times n} \underbrace{x}_{n\times 1})}_{1\times 1} = \frac{1}{\|u\|^2} (\vec{u}\,\vec{u}^T) x$$

$\alpha \in \mathbb{R}$ $\quad$ $\vec{x} \in \mathbb{R}^n$ $\quad$ $\underbrace{\alpha}_{1\times 1} \underbrace{\vec{x}}_{n\times 1}$ $\quad$ $\underbrace{\vec{x}}_{n\times 1} \underbrace{\alpha}_{1\times 1} \Rightarrow$ ضرب ماتریسی

1×1



$$x_p = P_S(x) = \frac{\bar{u} u^T}{\|u\|^2} x = \frac{\overbrace{\overset{n\times 1}{u}\ \overset{1\times n}{u^T}}^{n\times n}}{\underbrace{\underset{1\times n}{u^T}\ \underset{n\times 1}{u}}_{1\times 1}} x = Px$$

$$P = \frac{u u^T}{\|u\|^2}$$ projection matrix

$$\underset{1\times 1}{\alpha} \vec{x} = \alpha \underset{n\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} \stackrel{def}{=} \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix}$$

$$\underset{n\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} \underset{1\times 1}{[\alpha]} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix}$$

$$\underset{n\times 1}{x} \underbrace{(ABC)}_{1\times 1} = (xA)(BC)$$

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unit vector $v \Rightarrow \|v\| = 1$

$$P = \frac{u u^T}{\|u\|^2} \qquad \bar{u} = \frac{u}{\|u\|} \qquad \|\bar{u}\| = \sqrt{\frac{u^T u}{\|u\|^2}} = 1$$

$$P^T = \frac{u u^T}{\|u\|^2} = P \qquad \boxed{\langle y, Ax \rangle = \langle A^T y, x \rangle}$$

$$P P x = P x \quad \forall x \Rightarrow P P = P \qquad \frac{u \boxed{u^T u} u^T}{\|u\|^4} = \frac{u \|u\|^2 u^T}{\|u\|^4} = \frac{u u^T}{\|u\|^2}$$

idempotent function

MA

$$f(f(x)) = f(x)$$