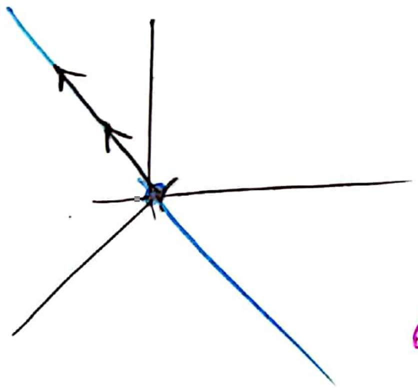
$$2x+3y-z=0$$
$$x-2y-4z=0$$

$x=2$, $y=-1$, $z=1$ $\qquad$ $x=4$, $y=-2$, $z=2$ $\qquad$ $x=2\alpha$, $y=-\alpha$, $z=\alpha$

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & -2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a solution, so is $\alpha \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for all $\alpha \in \mathbb{R}$.



$$A\vec{x} = \vec{0}$$ معادلات همگن

$A \in \mathbb{R}^{m\times n}$, $\vec{x} \in \mathbb{R}^n$, $\vec{0} \in \mathbb{R}^m$

Only the direction of $\vec{x}$ matters!

All Solutions to $A\vec{x}=\vec{0}$ $\equiv N(A)$

How to represent a linear subspace $S \subseteq \mathbb{R}^n$?

$\dim(S) = d$

1- A basis $V = \begin{bmatrix} v_1 & v_2 & \cdots & v_d \end{bmatrix} \equiv VH$

$H \in \mathbb{R}^{d\times d}$; invertible

$V' = VH$ is also a basis.

$$\underbrace{V^T V}_{d\times d} \vec{a} = V^T x \qquad V = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_d \\ | & | & & | \end{bmatrix}$$

$$\vec{a} = (V^T V)^{-1} V^T \vec{x} \qquad n\times d \qquad d \leq n$$

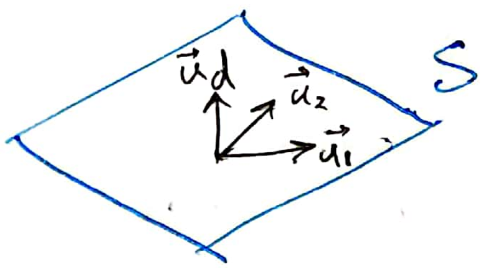
$$x_p = V\vec{a} = V(V^T V)^{-1} V^T \vec{x}$$

$$x_p = P x \qquad P = V(V^T V)^{-1} V^T$$

$$S = \cancel{\text{span}}\ C(V) \qquad P^T = P$$

$$PP = P$$

V full-column-rank



$\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_d$ a basis for S

$$U = [u_1, \ldots, u_d]$$

$$P = U(U^T U)^{-1} U^T$$

Assume that $\vec{u}_1; \vec{u}_2, \ldots, \vec{u}_d$ orthonormal

$$\vec{u}_i \perp \vec{u}_j \quad i \neq j \Rightarrow u_i^T u_j = 0 \quad i \neq j$$

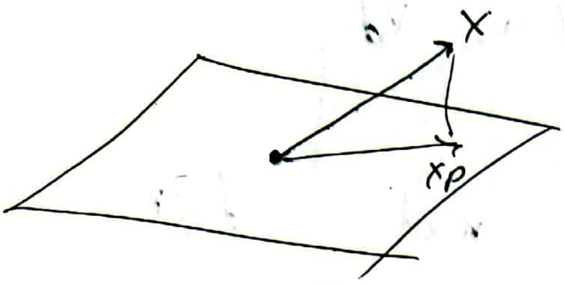
$$\|u_i\| = 1 \qquad \Rightarrow u_i^T u_j = 1 \quad i = j$$

$$U = \begin{bmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_d \\ | & | & & | \end{bmatrix} \qquad \underset{d\times n}{U^T U = \begin{bmatrix} — u_1^T — \\ — u_2^T — \\ \vdots \\ — u_d^T — \end{bmatrix}} \underset{n\times d}{\begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_d \\ | & | & & | \end{bmatrix}} = \underset{d\times d}{I_d}$$

$UU^T \in \mathbb{R}^{n\times n}$ $\quad \text{rank}(UU^T) = d \quad UU^T \neq I$ if $d < n$

$$P = U(U^T U)^{-1} U^T = UU^T = [u_1\ u_2 \cdots u_d] \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_d^T \end{bmatrix} = \sum_{i=1}^{d} \vec{u}_i \vec{u}_i^T$$

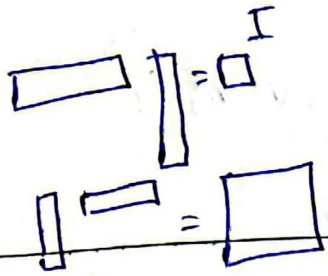
$\vec{u}_1, \vec{u}_2, \dots, \vec{u}_d$ orthonormal

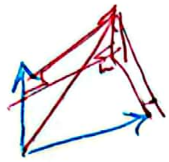


$$x_p = Px = \left(\sum_{i=1}^{d} u_i u_i^T\right) x$$

$$= \sum_{i=1}^{d} u_i u_i^T x = \sum_{i=1}^{d} \frac{u_i u_i^T}{\|u\|^2} \vec{x}$$

1





$$= \sum_{i=1}^{d} (u_i^T x)\, \vec{u}_i$$

$d = n \qquad P = UU^T$

$$\left.\begin{array}{l} U^T U = I \\ U \in \mathbb{R}^{n\times d} = \mathbb{R}^{n\times n} \\ \text{rank}(U) = n \end{array}\right\} \quad U^T = U^{-1} \Rightarrow UU^T = I$$

U is an orthogonal matrix یعنی

$\Rightarrow U \in \mathbb{R}^{n\times n}$ has n ~~orthogonal~~ orthonormal columns

$$U = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \qquad U = \begin{bmatrix} r_1^T \\ r_2^T \\ \vdots \\ r_n^T \end{bmatrix}$$

$$UU^T = I = \begin{bmatrix} r_1^T \\ r_2^T \\ \vdots \\ r_n \end{bmatrix} \begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix}$$

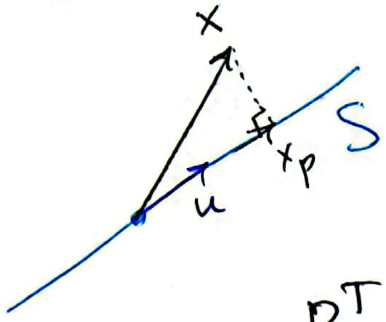
U has orthonormal rows

U ~~orthonormal~~ orthogonal $\qquad U^T U = UU^T = I$

idempotent function

$$f(f(x)) = f(x)$$

MA (VI)



$\phi \quad x_p = P_S(\vec{x}) = \boxed{\dfrac{uu^T}{\|u\|^2}} \, x$ MA9 (I)

$\vec{x} \in \mathbb{R}^n$

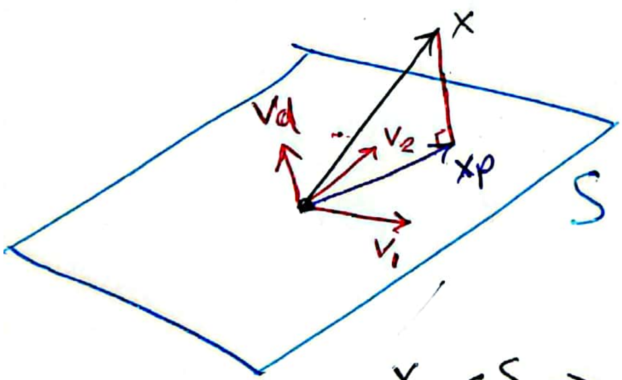
P projection matrix

$$P^T = P$$

$$PP = P$$

$$\text{rank}(P) = \text{rank}(uu^T) = \text{rank}\left(\begin{bmatrix} u_1\vec{u} & u_2\vec{u} & \cdots & u_n\vec{u} \end{bmatrix} \right)$$

$$= 1$$



$S \in \mathbb{R}^n \quad x \in \mathbb{R}^n$

$\dim(S) = d \leq n$

$\vec{V}_1, \vec{V}_2, \ldots, \vec{V}_d$ a basis for S

$$x_p \in S \Rightarrow x_p = \sum_{i=1}^{d} \alpha_i \vec{V}_i = \begin{bmatrix} | & | & & | \\ \vec{V}_1 & \vec{V}_2 & \cdots & \vec{V}_d \\ | & | & & | \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_d \end{bmatrix}$$

$\vec{a}$

$$x_p = V \vec{a}$$

$\in \mathbb{R}^n \quad \in \mathbb{R}^{n \times d} \quad \in \mathbb{R}^d$

$x - x_p \perp v_1 \Rightarrow \langle x - x_p, V_i \rangle = v_i^T(x - x_p) = 0 \Rightarrow v_i^T(x - Va) = 0$

$x - x_p \perp V_2$

$\vdots$

$x - x_p \perp V_d$

$$\begin{bmatrix} - V_1^T - \\ V_2^T - \\ \vdots \\ - V_d^T - \end{bmatrix} (x - Va) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow V^T(x - Va) = \vec{0}$$

$$V^T x - V^T V a = \vec{0}$$

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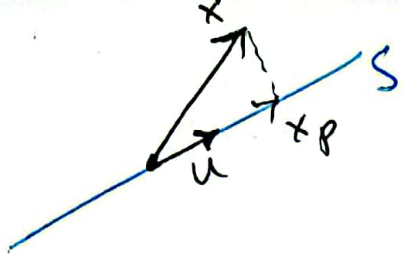
MA 8 V



$$x_p = P_S(x) = \frac{uu^T}{\|u\|^2}\, x = \frac{\overbrace{\underbrace{u}_{n\times 1}\ \underbrace{u^T}_{1\times n}}^{n\times n}}{\underbrace{\underbrace{u^T}_{1\times n}\ \underbrace{u}_{n\times 1}}_{1\times 1}}\, x = Px$$

$$P = \frac{uu^T}{\|u\|^2}$$ projection matrix

$$\alpha \vec{x} = \underset{1\times 1}{\alpha} \underset{n\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} \overset{def}{=} \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix}$$

$$\underset{n\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} \underset{1\times 1}{[\alpha]} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix}$$

$$\underset{n\times 1}{x}\ \underbrace{(ABC)}_{1\times 1} = (xA)(BC)$$

---

unit vector $v \Rightarrow \|v\| = 1$

$$\bar{u} = \frac{u}{\|u\|} \qquad \|\bar{u}\| = \sqrt{\frac{u^Tu}{\|u\|^2}} = 1$$

$$P = \frac{uu^T}{\|u\|^2}$$

$$P^T = \frac{uu^T}{\|u\|^2} = P$$

$$\boxed{\langle y, Ax\rangle = \langle A^Ty, x\rangle}$$

$$PPx = Px \quad \forall x \Rightarrow PP = P$$

$$\frac{u\boxed{u^Tu}u^T}{\|u\|^4} = \frac{u\|u\|^2u^T}{\|u\|^4} = \frac{uu^T}{\|u\|^2}$$

MA9 (V)

$$\underset{m\times n}{A}\ \underset{\mathbb{R}^n}{\vec{x}} = \underset{\mathbb{R}^m}{\vec{b}}$$

m equations

n unknowns

$m < n$ $\quad A = \begin{bmatrix} \quad & \quad & \quad \end{bmatrix}$ underdetermined

$m > n$ $\quad A = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$ overdetermined

---

$$A \vec{x} = \vec{Y}$$

$\vec{x}$ → unknown

$\vec{Y}$ → measurement

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \qquad y_i = \sum_{j=1}^{n} a_{ij} x_j$$

$$m \gg n$$

In reality $\quad y_i = \underbrace{\sum_{j=1}^{n} a_{ij} x_j}_{\tilde{y}_i} + \varepsilon_i$ → noise

$\tilde{y}_i$ → true y_i

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_m \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$\vec{\varepsilon}$ noise vector

$\vec{y} \notin C(A)$ $\qquad A\vec{x} + \vec{\varepsilon} = \vec{Y}$

unknown yet small

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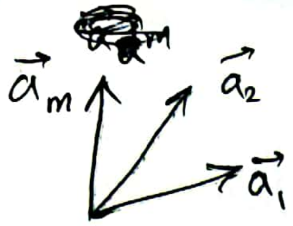
MA 9 (IV)



$\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m \in \mathbb{R}^n$

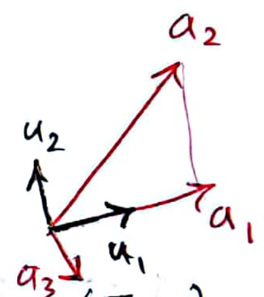
independent

$$A = \begin{bmatrix} | & | & & | \\ a_1, & a_2, & \ldots & a_m \\ | & | & & | \end{bmatrix} \Longrightarrow U = \begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix}$$

$\mathrm{span}(u_1, —, u_m) = \mathrm{span}(a_1 — a_m)$

$u_1, —, u_m$ orthonormal

$$u_1 = \frac{a_1}{\|a_1\|}$$



$\tilde{u}_1 = a_1$

$u_1 = \dfrac{\tilde{u}_1}{\|\tilde{u}_1\|}$

$\tilde{u}_2 = a_2 - u_1(u_1^T a_2)$

$u_2 = \dfrac{\tilde{u}_2}{\|u_2\|}$

$\tilde{u}_3 = a_3 - u_1 u_1^T a_3 - u_2 u_2^T a_3$

$u_3 = \dfrac{\tilde{u}_3}{\|\tilde{u}_3\|}$

$u_1^T u_3 = u_1^T a_3 - \underbrace{u_1^T u_1}_{1} u_1^T a_3 - \cancel{u_1^T u_2} u_2^T a_3$ $\cancel{0}$

$= u_1^T a_3 - u_1^T a_3 = 0$

$\vec{a}_1 = \alpha_{11} \vec{u}_1$

$\vec{a}_2 = \alpha_{12} \vec{u}_1 + \alpha_{22} \vec{u}_2$

$\vec{a}_3 = \alpha_{13} \vec{u}_1 + \alpha_{23} \vec{u}_2 + \alpha_{33} \vec{u}_3$

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} = \underset{n \times d}{\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}} \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ 0 & \alpha_{22} & \alpha_{23} \\ 0 & 0 & \alpha_{33} \end{bmatrix}$$

~~$A \in R$~~ $A \in [\vec{a}_1\ \vec{a}_2 \sim \vec{a}_n] \in \mathbb{R}^{n \times n}$ $\quad \mathrm{rank}(A) = n$

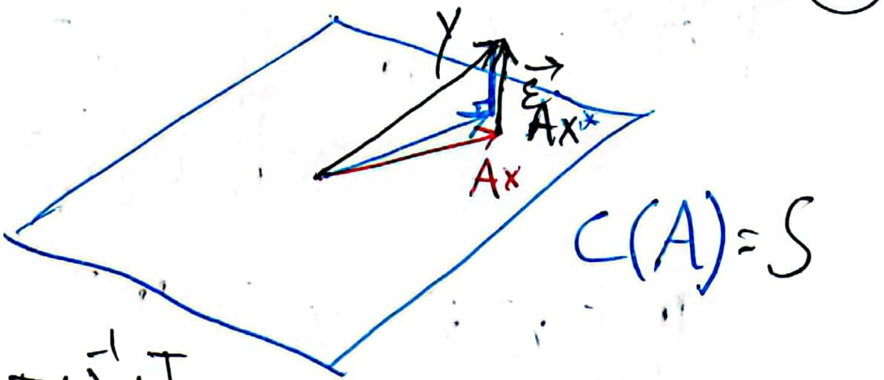
$$A = \underset{n \times n}{U} R = QR$$

$R \to$ upper-triangular

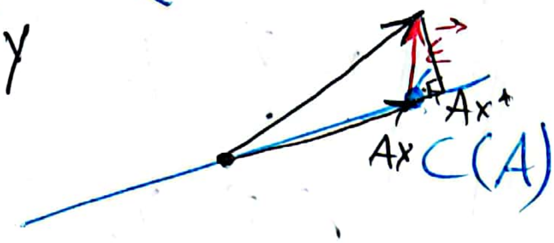
$Q \to$ orthogonal

QR decomposition

$$A\underline{x} + \varepsilon = y$$



$$Ax^* = P_S(y) = A(A^TA)^{-1}A^Ty$$



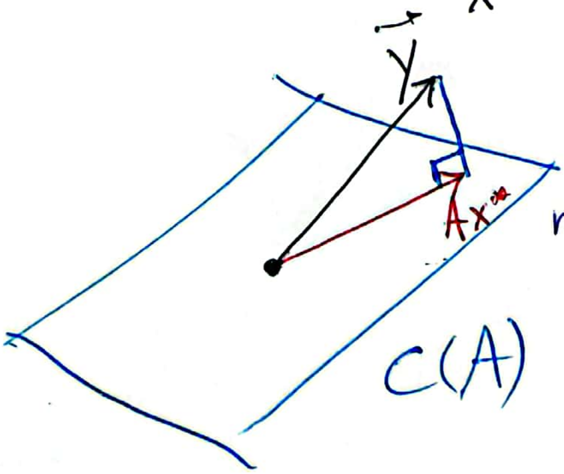
$$x^* = (A^TA)^{-1}A^Ty$$

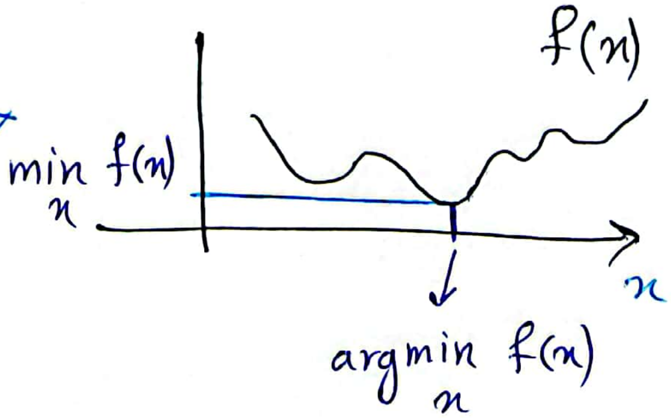
$$Ax = y \quad , \quad \vec{y} \notin C(A)$$

$$\nexists x \quad Ax = y \qquad \|\overbrace{\underset{m\times n}{A}\,\underset{n}{x} - \underset{m}{y}}^{\in \mathbb{R}^m}\|$$

$$x^* = \|A\backslash$$

$$x^* = \underset{\vec{x}}{\arg\min} \|A\vec{x} - \vec{y}\| = (A^TA)^{-1}A^T\vec{y}$$





$$\|A\vec{x}-\vec{y}\|^2 = \left\| \begin{bmatrix} r_1^T \\ r_2^T \\ \vdots \\ r_m^T \end{bmatrix} \vec{x} - \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \right\|^2$$

$$= \left\| \begin{bmatrix} r_1^T\vec{x} - y_1 \\ r_2^T\vec{x} - y_2 \\ \vdots \\ r_m^T\vec{x} - y_m \end{bmatrix} \right\|^2 = \underbrace{\sum_{i=1}^{m} \left(r_i^T\vec{x} - y_i\right)^2}_{\text{sum of squares}}$$

* $\min \|A\vec{x}-\vec{y}\|$ least squares
کمینه مربعات

$$x^* = \underset{}{\operatorname{argmin}} \|Ax-y\| = (A^TA)^{-1}A^T\vec{y}$$

→ least squares solution