PGM - Homework 1

Question 1: Consider a cylindrical water well with a radius of 1 meter. We drop stones from the top-centre of the well. But due to air flow inside the well the stones will not exactly land on the centre at the bottom of the well. The bottom of the well is a circular region represented as

$$C = \{(x, y) \,|\, x^2 + y^2 \leq 1\}. \tag{1}$$

Where the stones fall follows a probabilistic distribution represented by a PDF $f\colon C \mapsto \mathbb{R}$ with the following form:

$$f(x, y) = \alpha(1 - \sqrt{x^2 + y^2}) \tag{2}$$

1. What is the value of α? (Hint: You may use polar coordinates.)
2. What is the probability of a stone landing in the region

$$S = \{(x, y) \,|\, \frac{1}{4} \leq \sqrt{x^2 + y^2} \leq \frac{3}{4}\}. \tag{3}$$

Question 2: Consider the following joint probability mass function (PMF) $f_{I,N}\colon \{2, 3, 4\} \times \mathbb{N} \to [0, 1]$:

$$f_{I,N}(i, n) = P(I = i, N = n) = \alpha \left(\frac{1}{i}\right)^n \tag{4}$$

where $i \in \{2, 3, 4\}$ and $n \in \mathbb{N} = \{1, 2, 3, \cdots\}$.

1. Derive the value of α?
2. Obtain the formulae for the marginal distributions $f_I(i)$ and $f_N(n)$.
3. Obtain the formulae for the conditional distributions $f_I(i \mid N = n)$ and $f_N(n \mid I = i)$.
4. Are the random variables I and N independent? Why?

Question 3: Consider the following PMF $f_{M,N}\colon \{1,2,3\} \times \{1,2\}$:

m	n	$f_{M,N}(m,n)$
1	1	1/24
1	2	1/6
2	1	1/12
2	2	1/3
3	1	1/8
3	2	1/2

1. Derive $f_M(m)$, $f_N(n)$, $f_M(m \mid N = n)$ and $f_N(n \mid M = m)$.
2. Are M and N independent? Why?

Question 4: Consider the joint probability density function (PDF) $f_{X,Y}\colon \mathbb{R}^+ \times \mathbb{R}$, where $\mathbb{R}^+$ is the set of *nonnegative* real numbers, defined as

$$f_{X,Y}(x,y) = e^{-\alpha x - y^2}. \tag{5}$$

Notice that $x \geq 0$.

1. Derive the value of α?
2. Derive $f_X(x)$ and $f_Y(y)$.
3. Derive $f_X(x \mid Y = y)$ and $f_Y(y \mid X = x)$.
4. Are the random variables X and Y independent? Why?

Question 5: Answer the above questions for the PDF $f_{X,Y}\colon [1,4] \times \mathbb{R}$

$$f_{X,Y}(x,y) = e^{-\alpha x y^2}. \tag{6}$$

Notice that $x \in [1,4]$.

Hint:

$$\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}, \tag{7}$$

and, therefore,

$$\int_{-\infty}^{\infty} e^{-ay^2} dy = \sqrt{\frac{\pi}{a}}, \tag{8}$$

for any positive real number $a > 0$.

Question 6: Assume that the joint distribution $P(X, Y, Z)$ can be written as

$$P(X, Y, Z) = \phi(X, Z)\, \psi(Y, Z)$$

where $\phi(X, Z) > 0$ and $\psi(Y, Z) > 0$ for all values of X, Y, Z. Prove that

$$P(X, Y|Z) = P(X|Z)\, P(Y|Z)$$