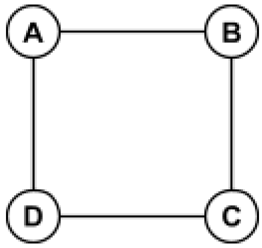
Probabilistic Graphical Models Final Exam - Spring 1402 (2023)	Instructor: B. Nasihatkon	دانشگاه صنعتی خواجه نصیرالدین طوسی K. N. TOOSI UNIVERSITY OF TECHNOLOGY
Name:	**ID:**	

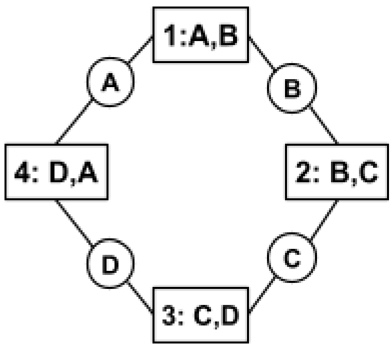
Question 1 - Loopy Max-sum message passing (35 points)



Consider the following MRF with the distribution
$P(A, B, C, D) = 1/Z \exp(\theta(A, B, C, D))$
where $\theta(A, B, C, D) = AB + 1(B = C) - CD + 1(D = A)$ and all the variables are binary $A, B, C, D \in \{0, 1\}$.

A) Calculate the max-marginal $M(A, B) = \max_{C} \max_{D} \theta(A, B, C, D)$. (10 points)

B) We intend to perform Loopy Max-Sum message passing using the following cluster graph. All the messages are equal to **zero** at the beginning. Let us just calculate the clock-wise messages. Compute $\delta_{1\to 2}(B)$, then $\delta_{2\to 3}(C)$, then $\delta_{3\to 4}(D)$, then $\delta_{4\to 1}(A)$. After that, compute $\delta_{1\to 2}(B)$ again. (20 points)



C) What will be the value of the max-sum message $\delta_{1\to 2}(B)$ after computing the messages n times? Do you think the algorithm will converge? (5 points)

Question 2 - Gibbs sampling (25 points)

Consider the MRF in Question 1:
$P(A, B, C, D) = 1/Z \exp(AB + 1(B = C) - CD + 1(D = A))$ with binary variables $A, B, C, D \in \{0, 1\}$. We want to take a sample from it using the Gibbs sampling algorithm. To perform a single MCMC transition

$A^t, B^t, C^t, D^t \to A^{t+1}, B^{t+1}, C^{t+1}, D^{t+1}$

we sample $A^{t+1}, B^{t+1}, C^{t+1}$, and D^{t+1} (in order) from the distributions $Q_A(A)$, $Q_B(B)$, $Q_C(C)$, and $Q_D(D)$, respectively. Derive $Q_A(A)$, $Q_B(B)$, $Q_C(C)$, and $Q_D(D)$. Simplify as much as possible. (be careful about using A^t or A^{t+1} etc., in the distributions).

Question 3 - Variable Elimination / MRF parameter estimation (40 points)

Consider the MRF $P(A, B, C, D) = 1/Z \exp(\alpha AB + 1(B = C) - CD + 1(D = A))$ where all the variables are binary $A, B, C, D \in \{0, 1\}$ and α is the only parameter.

A) Find the partition function Z as a function of α using variable elimination. The partition function will be in the form of $Z = p e^{\alpha} + q$. What are p and q? (10 points)

Hint: Write $\exp(\alpha AB + 1(B = C) - CD + 1(D = A))$ = $\exp(\alpha AB) \exp(1(B = C)) \exp(-CD) \exp(1(D = A))$, then apply variable elimination.

B) Write down the log-likelihood function considering the data below. To be concise, write in terms of p and q defined in part A. (14 points)

A^i	B^i	C^i	D^i
0	0	0	0
1	0	1	0
1	1	0	1
0	0	1	0

C) Find the maximum-likelihood solution for α by taking the setting derivative of log-likelihood equal to zero. Your solution can be in terms of p and q. (16 points)