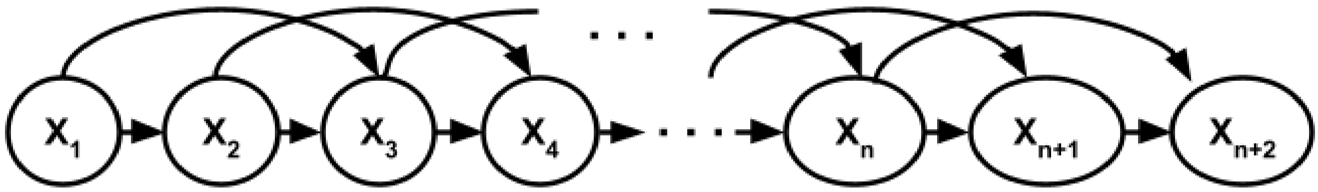
Probabilistic Graphical Models Midterm Exam - Spring 1403 (2024)	Instructor: B. Nasihatkon	دانشگاه صنعتی خواجه نصیرالدین طوسی K. N. TOOSI UNIVERSITY OF TECHNOLOGY
Name:	**ID:**	

Question 1 - Bayes Nets, Message Passing (60 points, 60 minutes)

Consider the following Bayesian Network on variables $X_1, X_2, ..., X_n, X_{n+1}, X_{n+2}$.



A) Write the joint distribution $p(X_1, X_2, ..., X_n, X_{n+1}, X_{n+2})$ in terms of the CPDs. **(5 points)**

B) Create a cluster tree (junction tree) with **exactly** n clusters, such that cluster ***i*** contains the variables X_i, X_{i+1}, X_{i+2}. Remember to draw the ***sepsets*** and assign factors (CPDs) to each cluster. **(10 points)**

C) Derive all the backward **sum-product** messages $\tau_{i \to i-1}$ for $i = 2, 3, ..., n$ for performing belief propagation. **(10 points)**

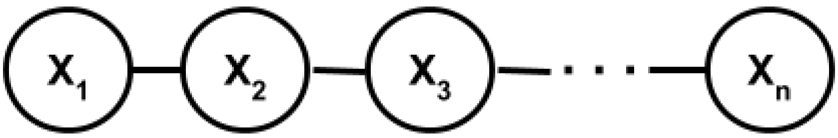
D) Now, assume that all the variables are binary ($X_i \in \{0, 1\}$) and the CPDs are
$p(X_1) = \exp(X_1) / (1 + e)$,
$p(X_2 \mid X_1) = \exp(X_1 X_2) / (1 + \exp(X_1))$,
$p(X_{i+2} \mid X_{i+1}, X_i) = \exp(X_i X_{i+1} X_{i+2}) / (1 + \exp(X_i X_{i+1}))$ for $i = 1, 2, \ldots, n$.
Derive the first forward message $\tau_{1 \rightarrow 2}$ as a function of the sepset. Simplify as much as you can. (15 points)

E) Following part (D), obtain the sum-product beliefs β_1 and β_2 for clusters 1 and 2. (10 points)

F) Using the CPDs introduced in part (D), compute the ***max-sum*** message $\sigma_{n \to n-1}$ from cluster n to cluster n-1. (10 points)

Question 2 - MRF / Variable Elimination (40 points, 35 minutes)

Consider the following MRF on binary variables $X_1, X_2, ..., X_n \in \{0, 1\}$.



The joint distribution is defined as $p(X_1, X_2, ..., X_n) = \frac{1}{Z} \prod_{i=1}^{n-1} \exp(X_i X_{i+1})$. We are to perform variable elimination in the order $X_1, X_2, ..., X_n$.

A) Show that the immediate factor created after eliminating X_i is in the form of $\tau_i(X_{i+1}) = a_i + b_i \exp(X_{i+1})$. To do this first derive the first factor $\tau_1(X_2)$. Then obtain τ_{i+1} from τ_i by eliminating X_{i+1} assuming that $\tau_i(X_{i+1}) = a_i + b_i \exp(X_{i+1})$. Find a recursive formula to obtain (a_{i+1}, b_{i+1}) from (a_i, b_i). **(20 points)**

B) Write the vector $[a_{i+1};\ b_{i+1}]$ as a 2 by 2 matrix times the vector $[a_i;\ b_i]$. **(5 points)**

C) Derive the partition function Z for $n = 4$ by eliminating all the variables. **(15 points)**