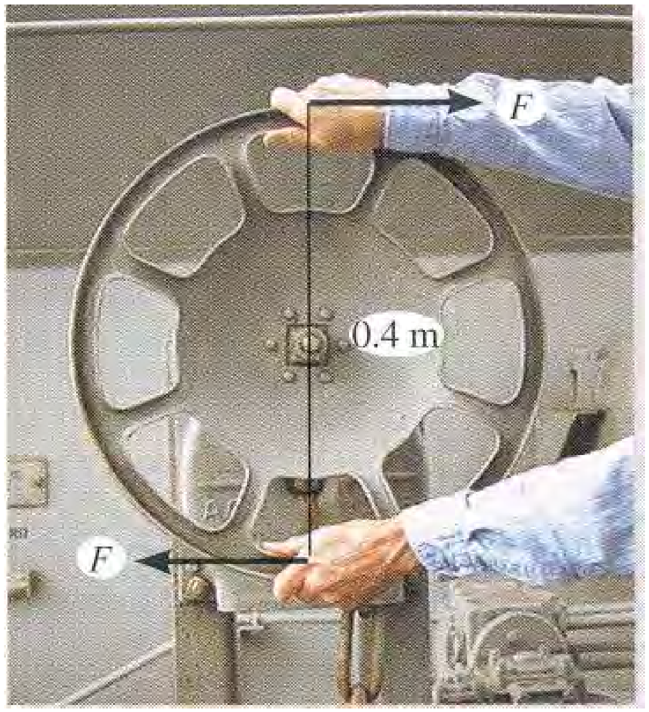
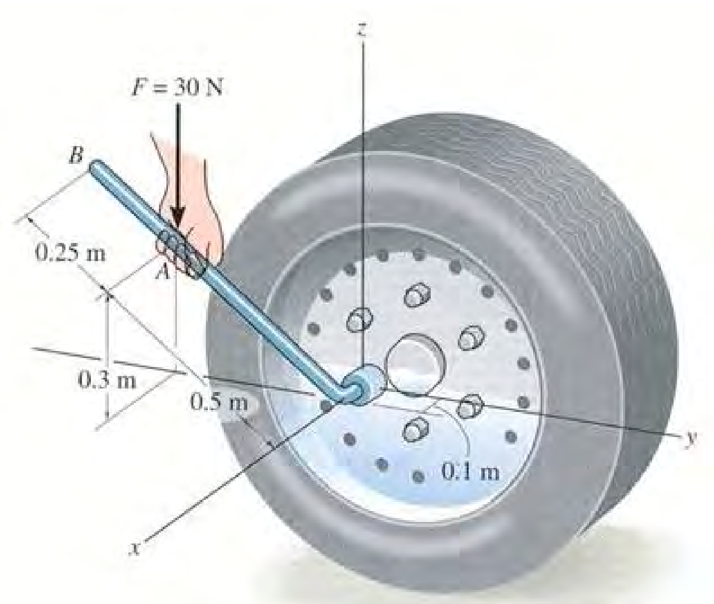
APPLICATIONS



What is the net effect of the two forces on the wheel?

APPLICATIONS (continued)



What is the effect of the 30 N force on the lug nut?

Introduction

- Treatment of a body as a single particle is not always possible. In general, the size of the body and the specific points of application of the forces must be considered.

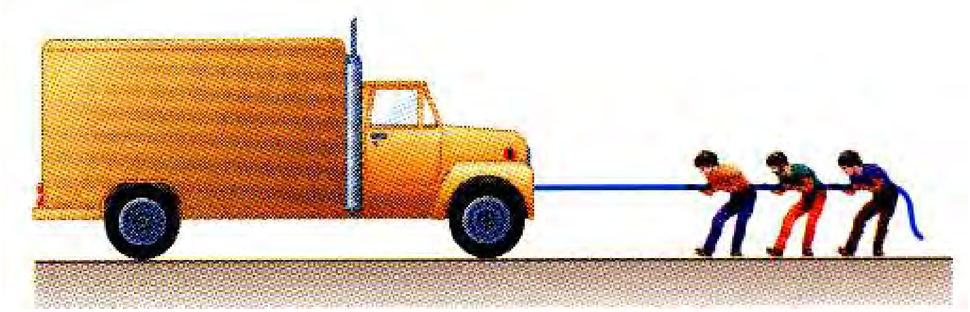
- Most bodies in elementary mechanics are assumed to be rigid, i.e., the actual deformations are small and do not affect the conditions of equilibrium or motion of the body.

- Current chapter describes the effect of forces exerted on a rigid body and how to replace a given system of forces with a simpler equivalent system.
  - moment of a force about a point
  - moment of a force about an axis
  - moment due to a couple

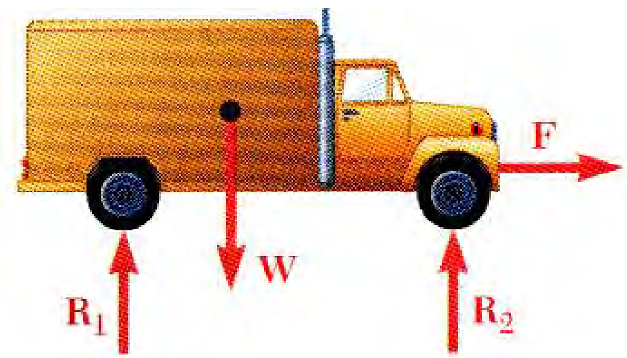
- Any system of forces acting on a rigid body can be replaced by an equivalent system consisting of one force acting at a given point and one couple.

External and Internal Forces

- Forces acting on rigid bodies are divided into two groups:
  - External forces
  - Internal forces



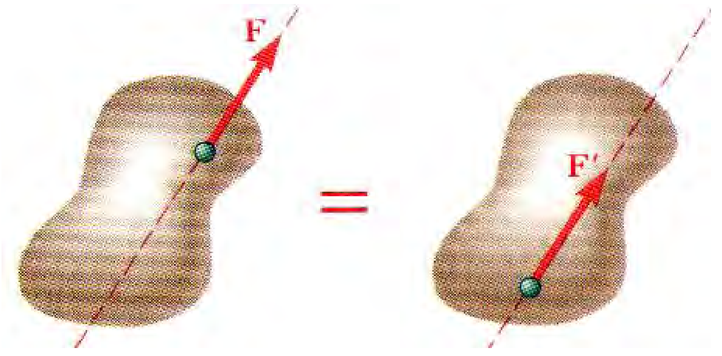
- External forces are shown in a free-body diagram.



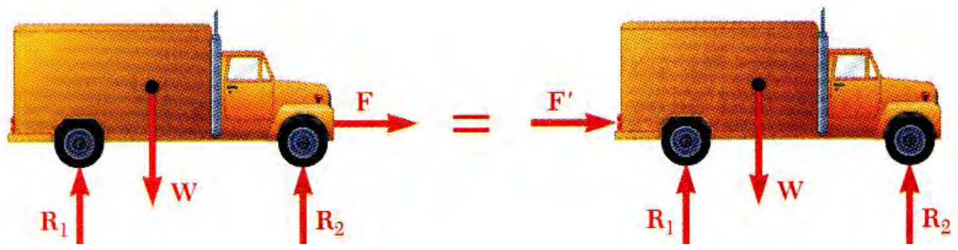
- If unopposed, each external force can impart a motion of translation or rotation, or both.

Principle of Transmissibility: Equivalent Forces

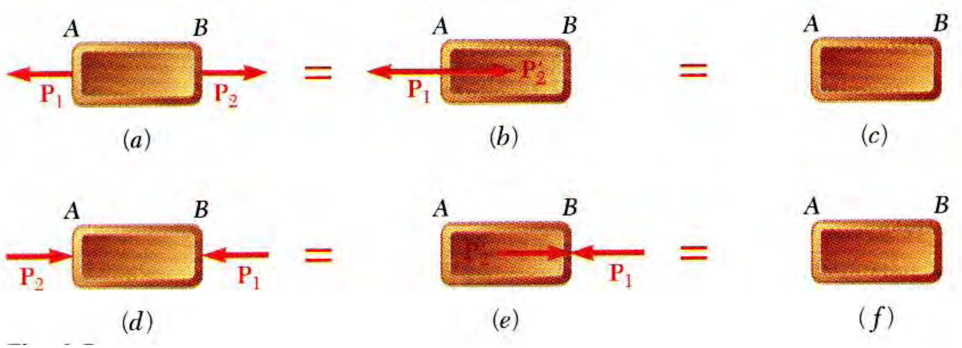
- *Principle of Transmissibility* - Conditions of equilibrium or motion are not affected by *transmitting* a force along its line of action. NOTE: **F** and **F'** are equivalent forces.



- Moving the point of application of the force **F** to the rear bumper does not affect the motion or the other forces acting on the truck.



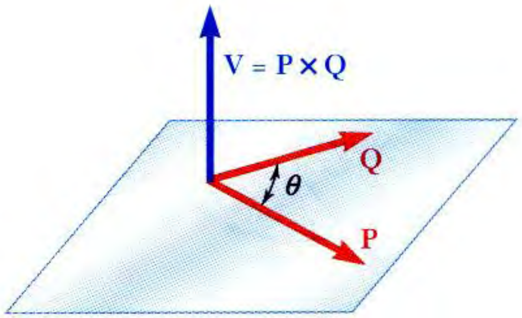
- Principle of transmissibility may not always apply in determining internal forces and deformations.



Vector Product of Two Vectors

- Concept of the moment of a force about a point is more easily understood through applications of the *vector product* or *cross product*.

- Vector product of two vectors $\boldsymbol{P}$ and $\boldsymbol{Q}$ is defined as the vector $\boldsymbol{V}$ which satisfies the following conditions:
    1. Line of action of $\boldsymbol{V}$ is perpendicular to plane containing $\boldsymbol{P}$ and $\boldsymbol{Q}$.
    2. Magnitude of $\boldsymbol{V}$ is $V = PQ\sin\theta$
    3. Direction of $\boldsymbol{V}$ is obtained from the right-hand rule.



(a)

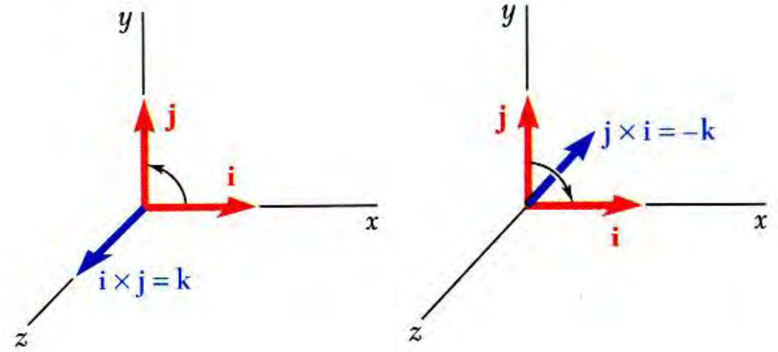
(b)

- Vector products:
    - are not commutative, $\boldsymbol{Q}\times\boldsymbol{P} = -(\boldsymbol{P}\times\boldsymbol{Q})$
    - are distributive, $\boldsymbol{P}\times(\boldsymbol{Q}_1+\boldsymbol{Q}_2) = \boldsymbol{P}\times\boldsymbol{Q}_1 + \boldsymbol{P}\times\boldsymbol{Q}_2$
    - are not associative, $(\boldsymbol{P}\times\boldsymbol{Q})\times\boldsymbol{S} \neq \boldsymbol{P}\times(\boldsymbol{Q}\times\boldsymbol{S})$

Vector Products: Rectangular Components

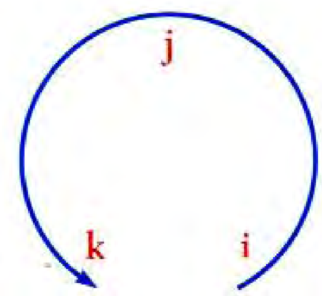
- Vector products of Cartesian unit vectors,

$$\vec{i}\times\vec{i}=0 \qquad \vec{j}\times\vec{i}=-\vec{k} \qquad \vec{k}\times\vec{i}=\vec{j}$$
$$\vec{i}\times\vec{j}=\vec{k} \qquad \vec{j}\times\vec{j}=0 \qquad \vec{k}\times\vec{j}=-\vec{i}$$
$$\vec{i}\times\vec{k}=-\vec{j} \qquad \vec{j}\times\vec{k}=\vec{i} \qquad \vec{k}\times\vec{k}=0$$



- Vector products in terms of rectangular coordinates

$$\vec{V}=\left(P_x\vec{i}+P_y\vec{j}+P_z\vec{k}\right)\times\left(Q_x\vec{i}+Q_y\vec{j}+Q_z\vec{k}\right)$$
$$=\left(P_yQ_z-P_zQ_y\right)\vec{i}+\left(P_zQ_x-P_xQ_z\right)\vec{j}$$
$$+\left(P_xQ_y-P_yQ_x\right)\vec{k}$$
$$=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ P_x & P_y & P_z\\ Q_x & Q_y & Q_z\end{vmatrix}$$



Moment of a Force About a Point

- A force vector is defined by its magnitude and direction. Its effect on the rigid body also depends on it point of application.

- The *moment* of ***F*** about *O* is defined as

$$\boldsymbol{M_O} = \boldsymbol{r} \times \boldsymbol{F}$$

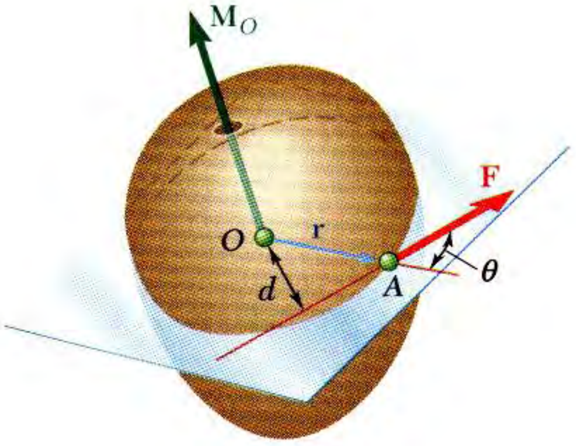
- The moment vector $\boldsymbol{M_O}$ is perpendicular to the plane containing *O* and the force ***F***.

- Magnitude of $\boldsymbol{M_O}$ measures the tendency of the force to cause rotation of the body about an axis along $\boldsymbol{M_O}$.

$$M_O = rF \sin\theta = Fd$$

  The sense of the moment may be determined by the right-hand rule.

- Any force ***F'*** that has the same magnitude and direction as ***F***, is *equivalent* if it also has the same line of action and therefore, produces the same moment.

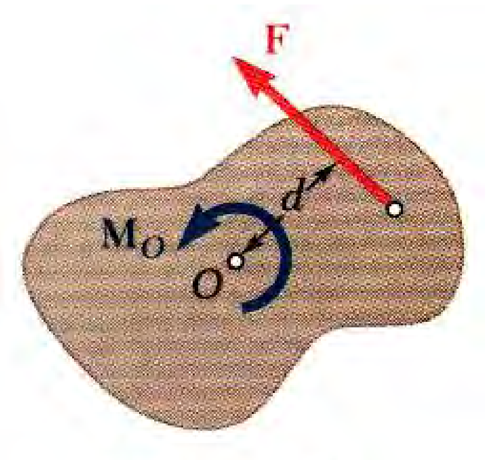


(a)

(b)

Moment of a Force About a Point

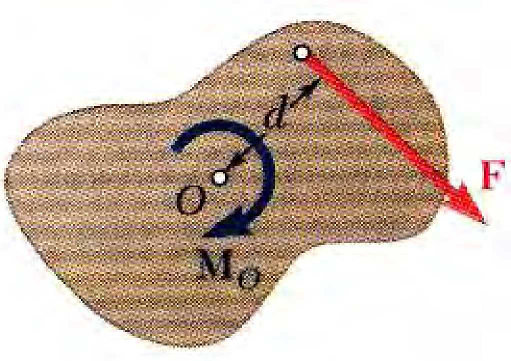
- *Two-dimensional structures* have length and breadth but negligible depth and are subjected to forces contained in the plane of the structure.



(a) $M_O = +Fd$

- The plane of the structure contains the point *O* and the force $\boldsymbol{F}$. $\boldsymbol{M_O}$, the moment of the force about *O* is perpendicular to the plane.

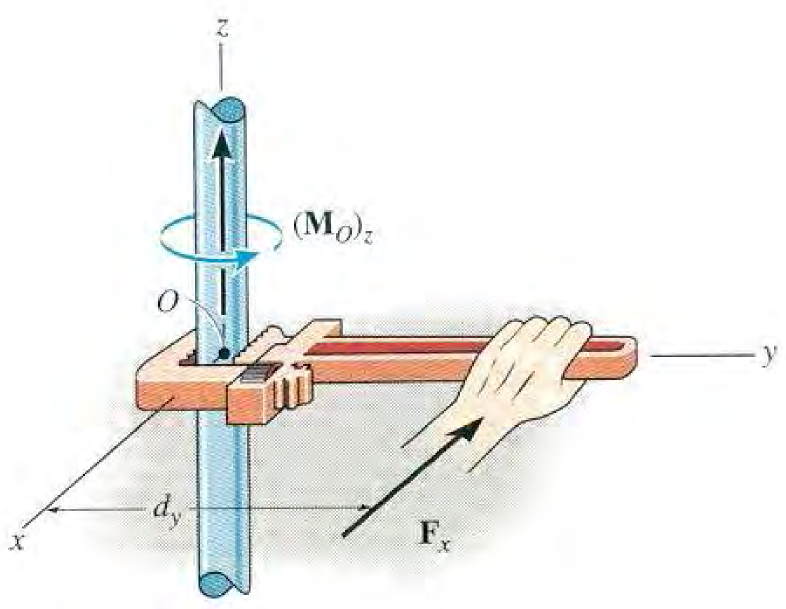
- If the force tends to rotate the structure clockwise, the sense of the moment vector is out of the plane of the structure and the magnitude of the moment is positive.



(b) $M_O = -Fd$

- If the force tends to rotate the structure counterclockwise, the sense of the moment vector is into the plane of the structure and the magnitude of the moment is negative.

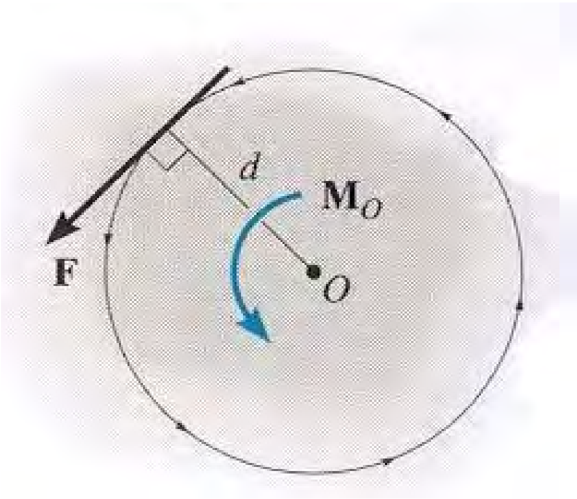
MOMENT IN 2-D



The moment of a force about a point provides a measure of the tendency for rotation (sometimes called a torque).

MOMENT IN 2-D (continued)

In the 2-D case, the magnitude of the moment is
$M_o = F\,d$

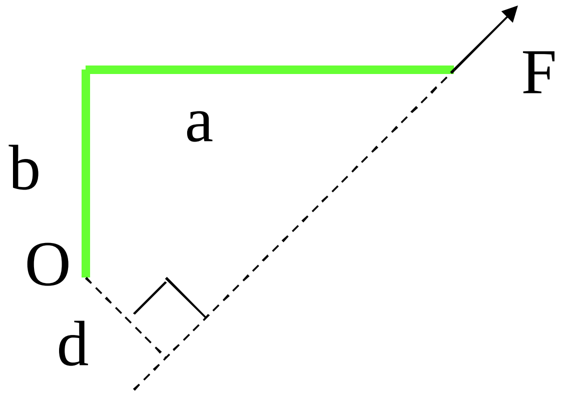


As shown, d is the *perpendicular* distance from point O to the line of action of the force.

In 2-D, the direction of M_O is either clockwise or

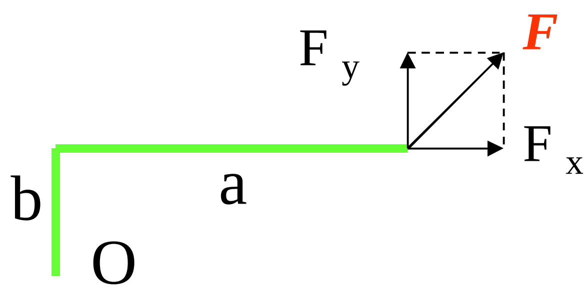
counter-clockwise depending on the tendency for rotation.

MOMENT IN 2-D (continued)



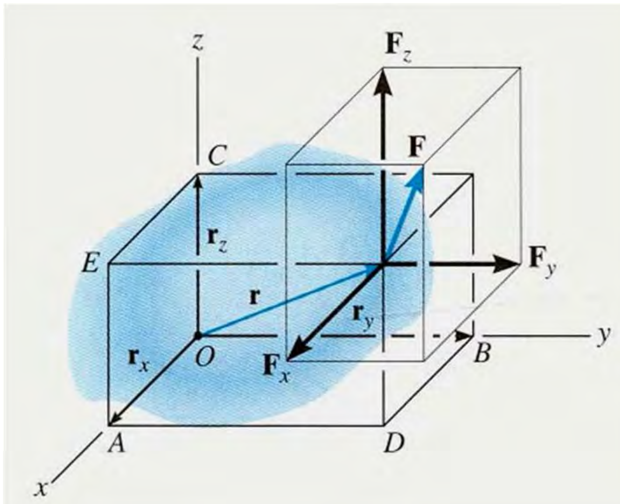
For example, $M_O = F\,d$ and the direction is counter-clockwise.

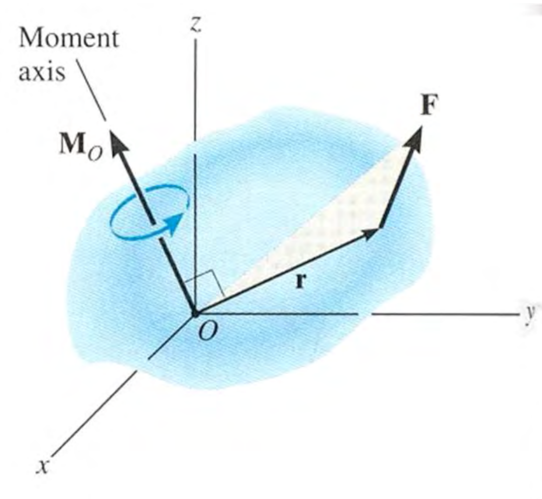
Often it is easier to determine M_O by using the components of ***F*** as shown.



Using this approach, $M_O = (F_Y\,a) - (F_X\,b)$. Note the different signs on the terms! The typical sign convention for a moment in 2-D is that counter-clockwise is considered positive. We can determine the direction of rotation by imagining the body pinned at O and deciding which way the body would rotate because of the force.

MOMENT IN 3-D (Vector formulation Section 4.3)



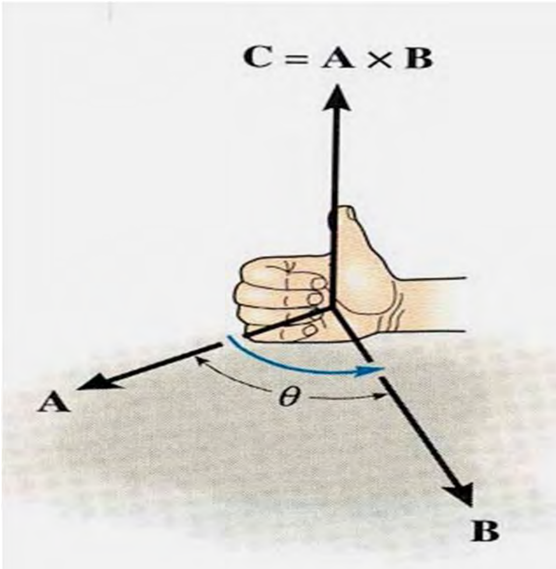


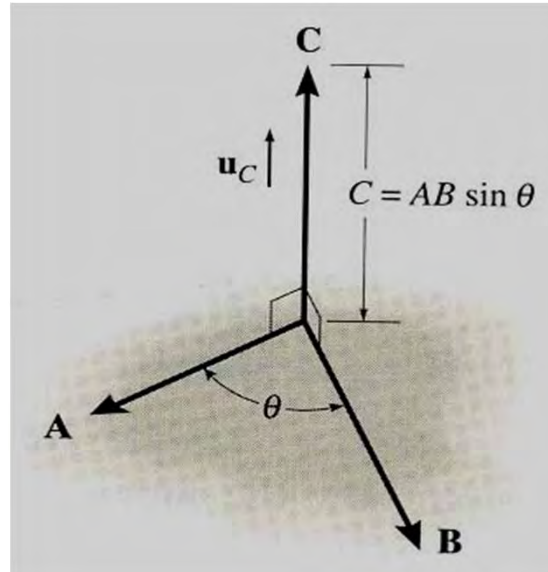
Moments in 3-D can be calculated using scalar (2-D) approach but it can be difficult and time consuming. Thus, it is often easier to use a mathematical approach called the vector cross product.

Using the vector cross product, $\boldsymbol{M_O} = \boldsymbol{r} \times \boldsymbol{F}$.

Here $\boldsymbol{r}$ is the position vector from point O to <u>any point</u> on the line of action of $\boldsymbol{F}$.

CROSS PRODUCT





In general, the cross product of two vectors ***A*** and ***B*** results in another vector ***C*** , i.e., $\boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B}$. The magnitude and direction of the resulting vector can be written as

$$\boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B} = A\, B \sin\theta\ \lambda_C$$

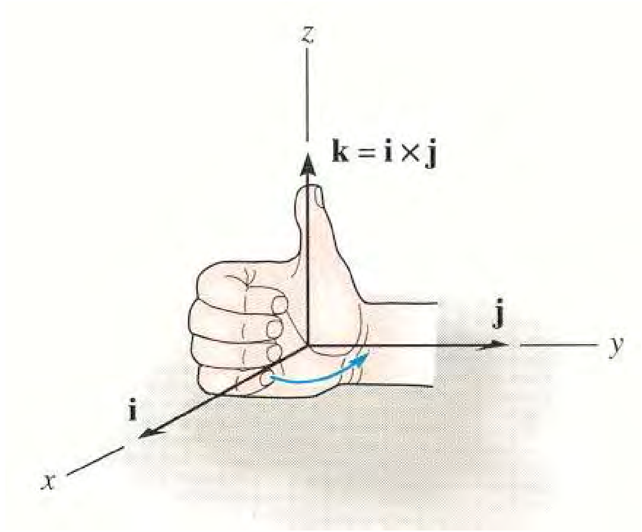
Here λ_C is the unit vector perpendicular to both A and B vectors as shown (or to the plane containing the A and B vectors).

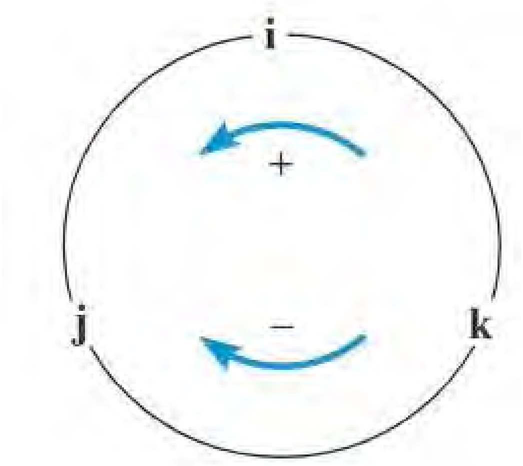
CROSS PRODUCT

The right hand rule is a useful tool for determining the direction of the vector resulting from a cross product.

For example: $\boldsymbol{i \times j = k}$

Note that a vector crossed into itself is zero, e.g., $\boldsymbol{i \times i = 0}$



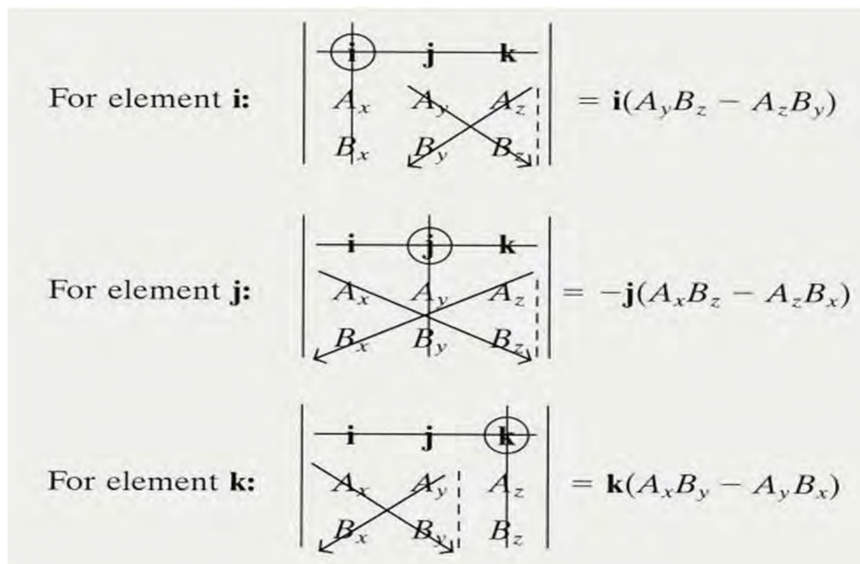


CROSS PRODUCT (continued)

Of even more utility, the cross product can be written as

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Each component can be determined using 2 × 2 determinants.



MOMENT IN 3-D (continued)

So, using the cross product, a moment can be expressed as

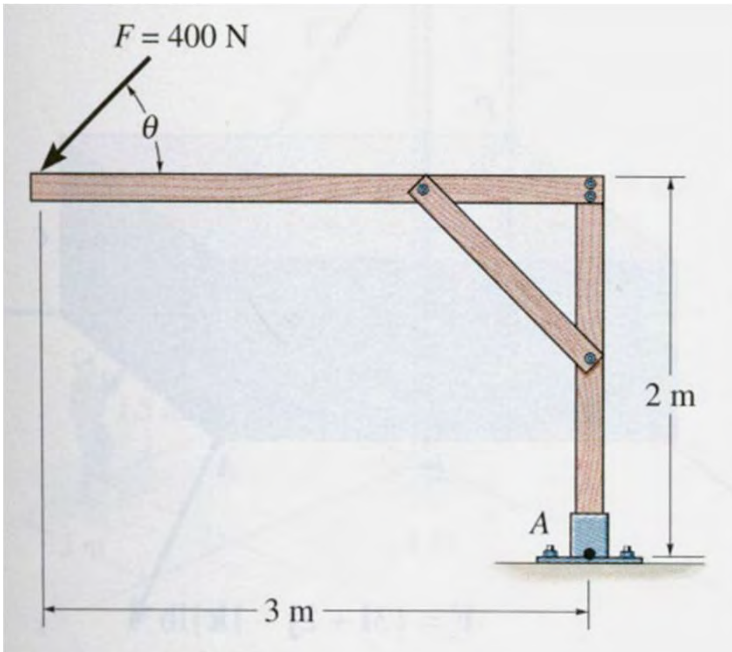
$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

By expanding the above equation using 2 × 2 determinants (see Section 4.2), we get (sample units are N - m or lb - ft)

$$\boldsymbol{M_O} = (r_y\ F_Z - r_Z\ F_y)\ \boldsymbol{i} - (r_x F_z - r_z F_x)\ \boldsymbol{j} + (r_x F_y - r_y F_x)\ \boldsymbol{k}$$

The physical meaning of the above equation becomes evident by considering the force components separately and using a 2-D formulation.

EXAMPLE 1



Given: A 400 N force is applied to the frame and $\theta = 20°$.

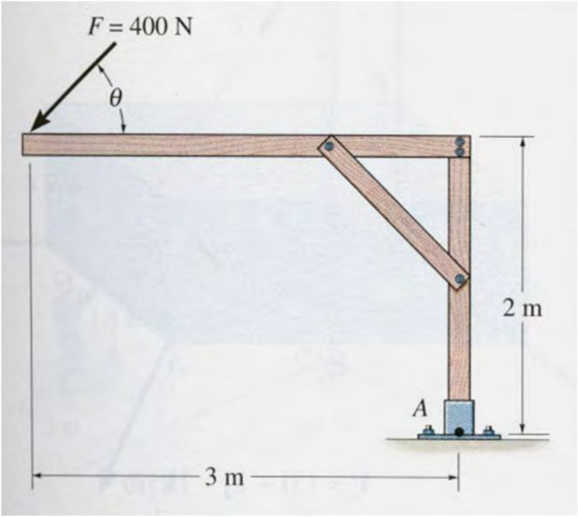
Find: The moment of the force at A.

Plan:

1) Resolve the force along x and y axes.

2) Determine M_A using scalar analysis.

EXAMPLE 1 (continued)



Solution

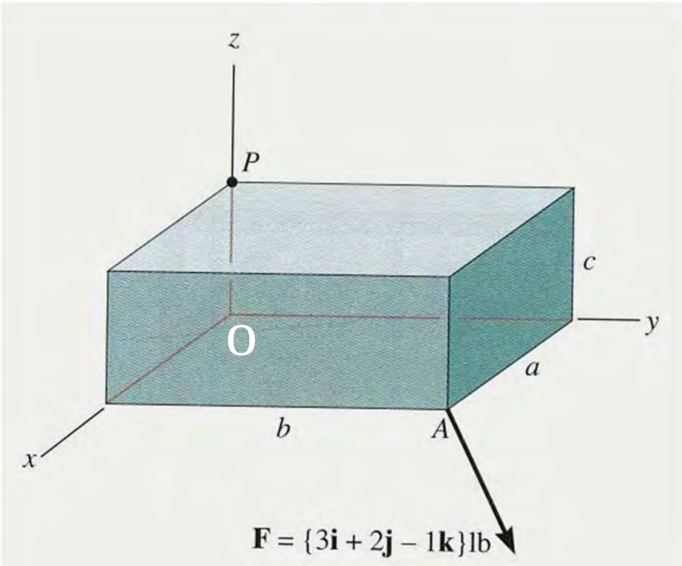
$$+\uparrow F_y = -400 \cos 20° \text{ N}$$

$$+\rightarrow F_x = -400 \sin 20° \text{ N}$$

$$\circlearrowleft + M_A = \{(400 \cos 20°)(2) + (400 \sin 20°)(3)\} \text{ N·m}$$

$$= 1160 \text{ N·m}$$

EXAMPLE 2



Given: a = 3 in, b = 6 in and c = 2 in.

Find: Moment of ***F*** about point O.

Plan:

1) Find $\boldsymbol{r}_{OA}$.

2) Determine $\boldsymbol{M}_O = \boldsymbol{r}_{OA} \times \boldsymbol{F}$.

Solution $\boldsymbol{r}_{OA} = \{3\,\boldsymbol{i} + 6\,\boldsymbol{j} - 0\,\boldsymbol{k}\}$ in

$$M_O = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 6 & 0 \\ 3 & 2 & -1 \end{vmatrix} = [\{6(-1) - 0(2)\}\,\boldsymbol{i} - \{3(-1) - 0(3)\}\,\boldsymbol{j} + \{3(2) - 6(3)\}\,\boldsymbol{k}]\ \text{lb}\cdot\text{in}$$

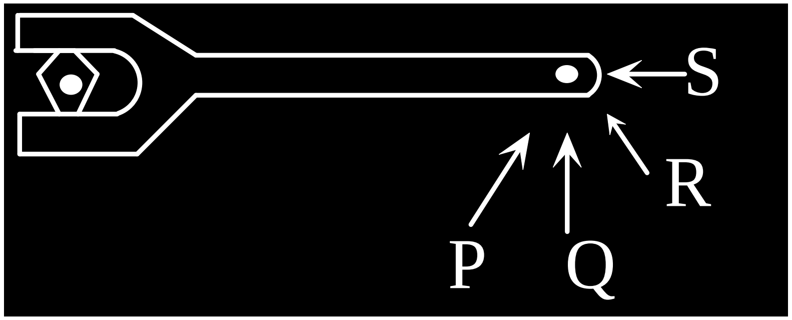
$$= \{-6\,\boldsymbol{i} + 3\,\boldsymbol{j} - 12\,\boldsymbol{k}\}\ \text{lb}\cdot\text{in}$$

CONCEPT QUIZ

1. If a force of magnitude F can be applied in four different 2-D configurations (P,Q,R, & S), select the cases resulting in the maximum and minimum torque values on the nut. (Max, Min).

A) (Q, P) B) (R, S)

C) (P, R) D) (Q, S)

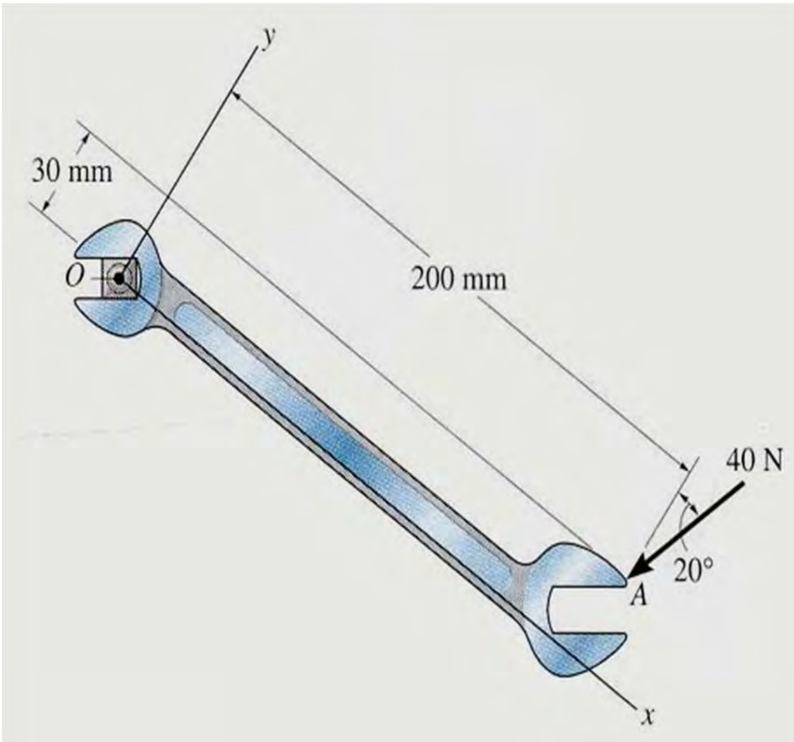


2. If $\mathbf{M} = \mathbf{r} \times \mathbf{F}$, then what will be the value of $\mathbf{M} \bullet \mathbf{r}$?

A) 0 B) 1

C) r^2 F D) None of the above.

GROUP PROBLEM SOLVING



Given: A 40 N force is applied to the wrench.

Find: The moment of the force at O.

Plan: 1) Resolve the force along x and y axes.

2) Determine M_O using scalar analysis.

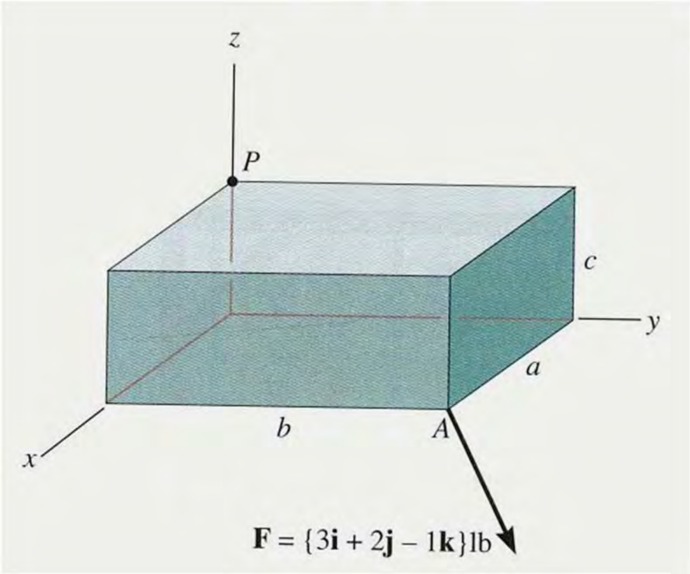
Solution: $+\uparrow F_y = -\,40 \cos 20° \text{ N}$

$+\rightarrow F_x = -\,40 \sin 20° \text{ N}$

$\circlearrowleft + \quad M_O = \{-(40 \cos 20°)(200) + (40 \sin 20°)(30)\} \text{N·mm}$

$= -7107 \text{ N·mm} = -\,7.11 \text{ N·m}$

GROUP PROBLEM SOLVING



Given: a = 3 in , b = 6 in and c = 2 in

Find: Moment of F about point P

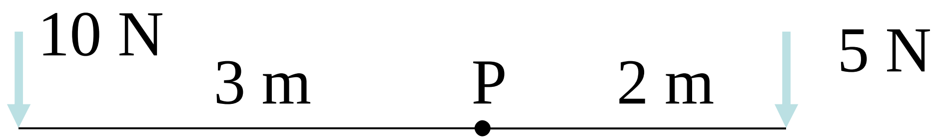
Plan: 1) Find $\mathbf{r}_{PA}$.

2) Determine $\mathbf{M}_P = \mathbf{r}_{PA} \times \mathbf{F}$

Solution: $\mathbf{r}_{PA} = \{ 3\,\mathbf{i} + 6\,\mathbf{j} - 2\,\mathbf{k} \}$ in

$$\mathbf{M}_P = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 6 & -2 \\ 3 & 2 & -1 \end{vmatrix} = \{ -2\,\mathbf{i} - 3\,\mathbf{j} - 12\,\mathbf{k} \} \text{ lb} \cdot \text{in}$$

ATTENTION QUIZ



1. Using the CCW direction as positive, the net moment of the two forces about point P is

A) 10 N ·m B) 20 N ·m C) - 20 N ·m

D) 40 N ·m E) - 40 N ·m

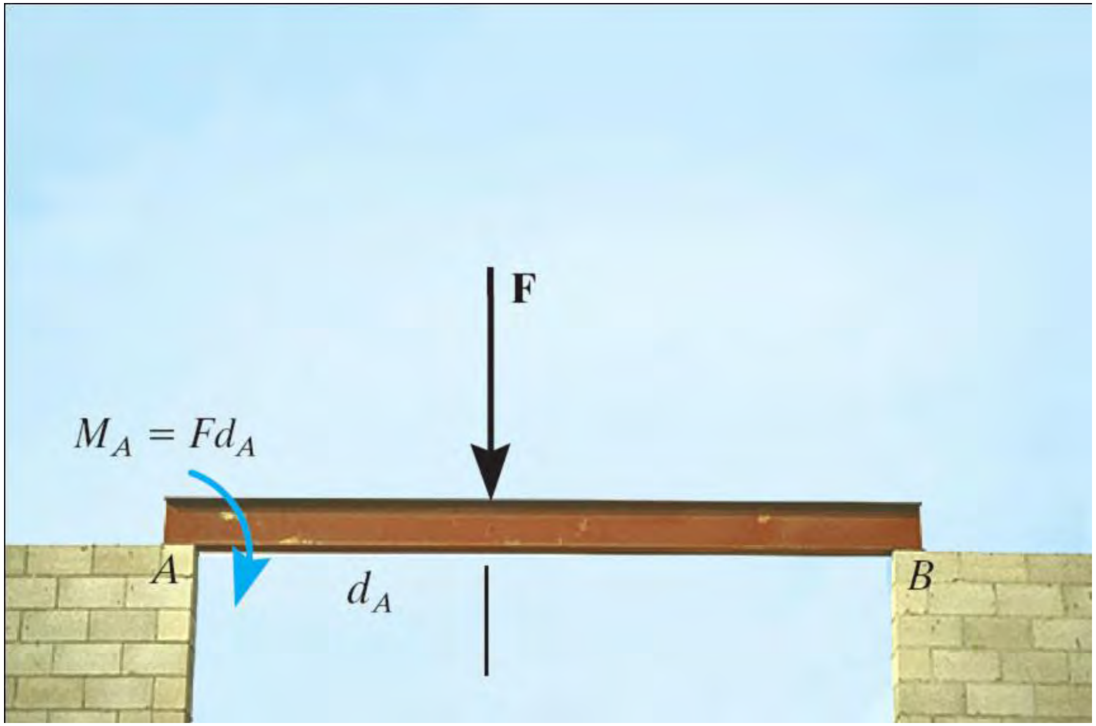
2. If ***r*** = { 5 ***j*** } m and ***F*** = { 10 ***k*** } N, the moment

r x F equals { ________ } N·m.

A) 50 ***i*** B) 50 ***j*** C) –50 ***i***

D) – 50 ***j*** E) 0

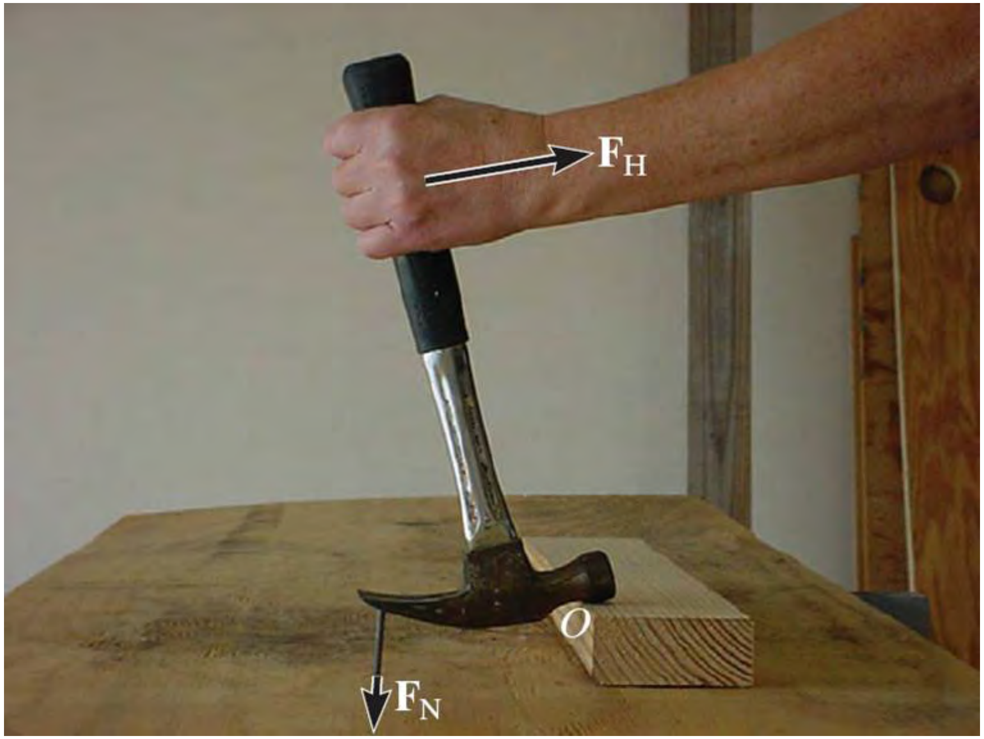
APPLICATIONS



Beams are often used to bridge gaps in walls. We have to know what the effect of the force on the beam will have on the supports of the beam.

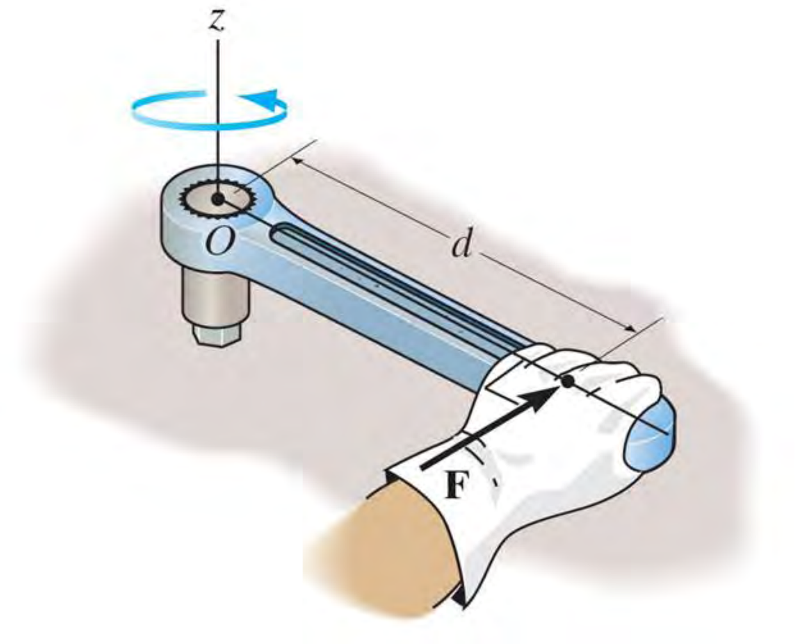
What do you think is happening at points A and B?

APPLICATIONS (continued)



Carpenters often use a hammer in this way to pull a stubborn nail. Through what sort of action does the force F_H at the handle pull the nail? How can you mathematically model the effect of force F_H at point O?

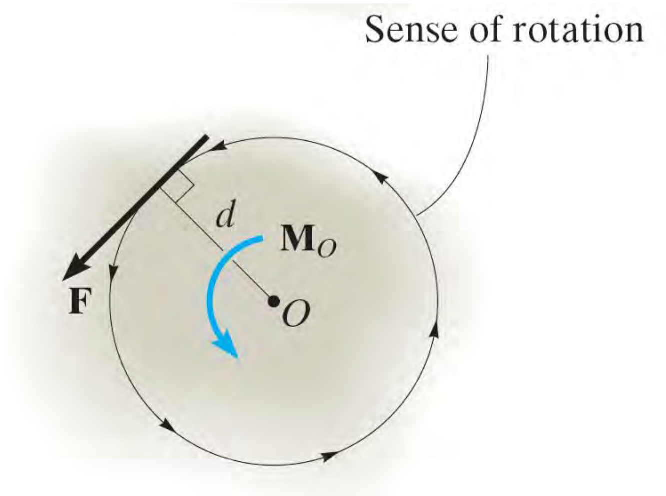
MOMENT OF A FORCE - SCALAR FORMULATION
(Section 4.1)



The moment of a force about a point provides a measure of the tendency for rotation (sometimes called a torque).

MOMENT OF A FORCE - SCALAR FORMULATION
(continued)

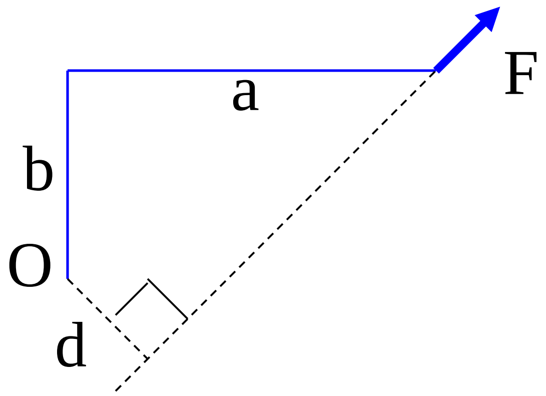
In a 2-D case, the magnitude of the moment is $M_o = F\,d$



As shown, d is the perpendicular distance from point O to the line of action of the force.

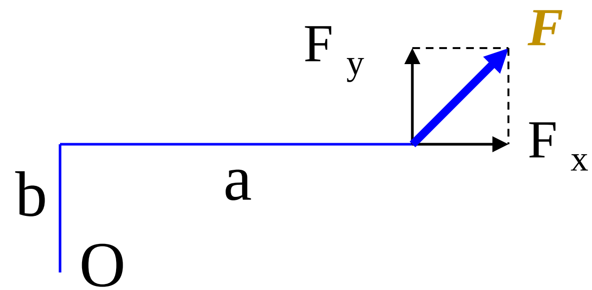
In 2-D, the direction of M_O is either clockwise (CW) or counter-clockwise (CCW), depending on the tendency for rotation.

MOMENT OF A FORCE - SCALAR FORMULATION
(continued)



For example, M_O = F d and the direction is counter-clockwise.

Often it is easier to determine M_O by using the components of ***F*** as shown.



Then $M_O = (F_Y\ a) - (F_X\ b)$. Note the different signs on the terms! The typical sign convention for a moment in 2-D is that counter-clockwise is considered positive. We can determine the direction of rotation by imagining the body pinned at O and deciding which way the body would rotate because of the force.

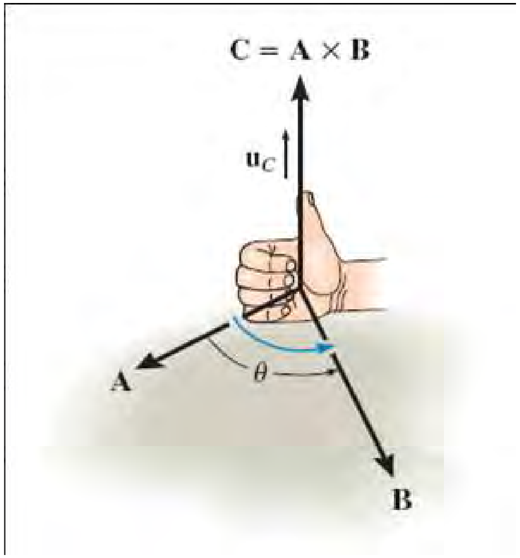
VECTOR CROSS PRODUCT (Section 4.2)

While finding the moment of a force in 2-D is straightforward when you know the perpendicular distance d, finding the perpendicular distances can be hard—especially when you are working with forces in three dimensions.

So a more general approach to finding the moment of a force exists. This more general approach is usually used when dealing with three dimensional forces but can be used in the two dimensional case as well.

This more general method of finding the moment of a force uses a vector operation called the cross product of two vectors.

CROSS PRODUCT (Section 4.2)



In general, the cross product of two vectors ***A*** and ***B*** results in another vector, ***C*** , i.e., $\mathbf{C} = \mathbf{A} \times \mathbf{B}$. The magnitude and direction of the resulting vector can be written as

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = A\, B \sin\theta\; \mathbf{u}_C$$

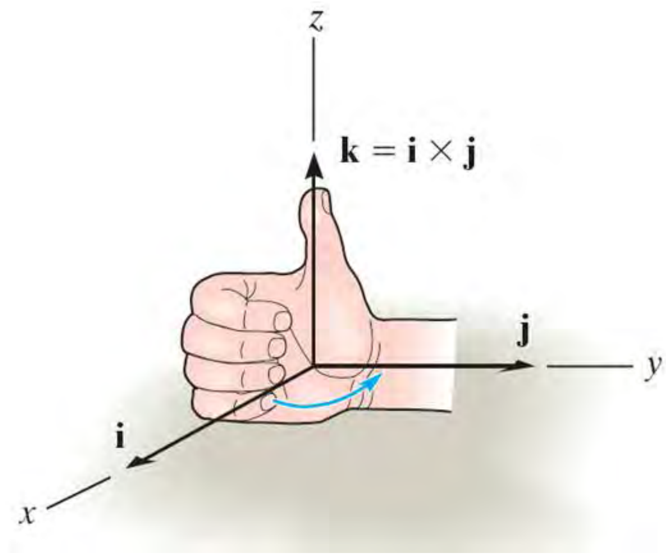
As shown, $\mathbf{u}_C$ is the unit vector perpendicular to both A and B vectors (or to the plane containing the A and B vectors).

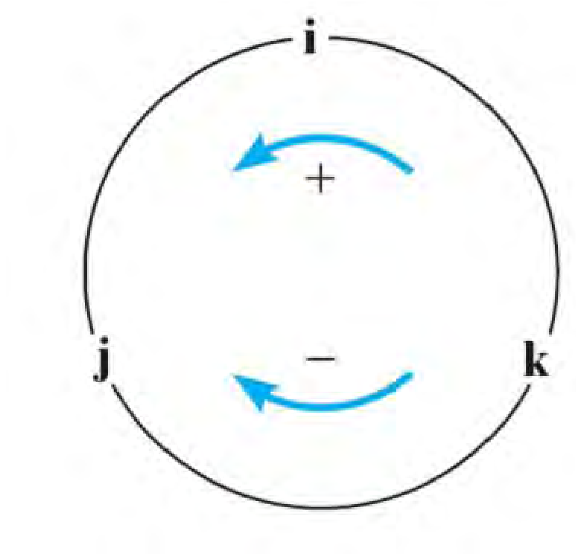
CROSS PRODUCT (continued)

The right-hand rule is a useful tool for determining the direction of the vector resulting from a cross product.

For example: $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$

Note that a vector crossed into itself is zero, e.g., $\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{0}$



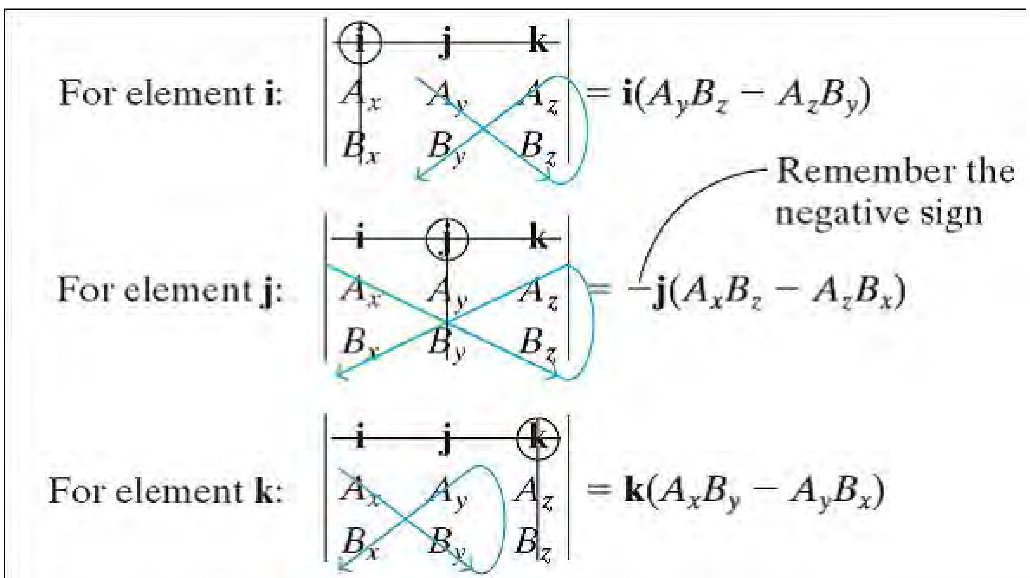


CROSS PRODUCT (continued)

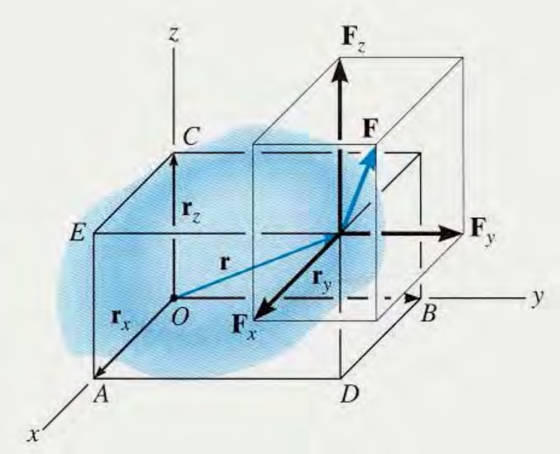
Also, the cross product can be written as a determinant.

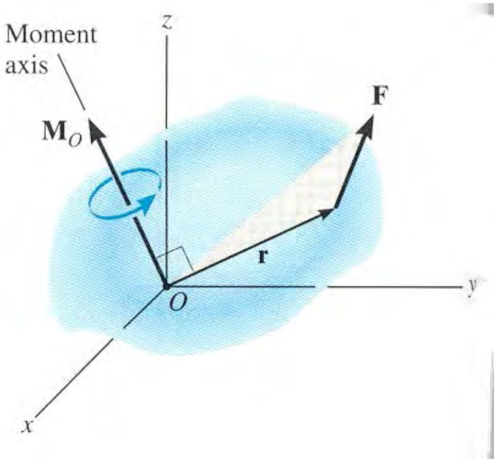
$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Each component can be determined using 2 × 2 determinants.



MOMENT OF A FORCE – VECTOR FORMULATION (Section 4.3)





Moments in 3-D can be calculated using scalar (2-D) approach, but it can be difficult and time consuming. Thus, it is often easier to use a mathematical approach called the vector cross product.

Using the vector cross product, $\boldsymbol{M_O} = \boldsymbol{r} \times \boldsymbol{F}$.

Here $\boldsymbol{r}$ is the position vector from point O to any point on the line of action of $\boldsymbol{F}$.

MOMENT OF A FORCE – VECTOR FORMULATION
(continued)

So, using the cross product, a moment can be expressed as

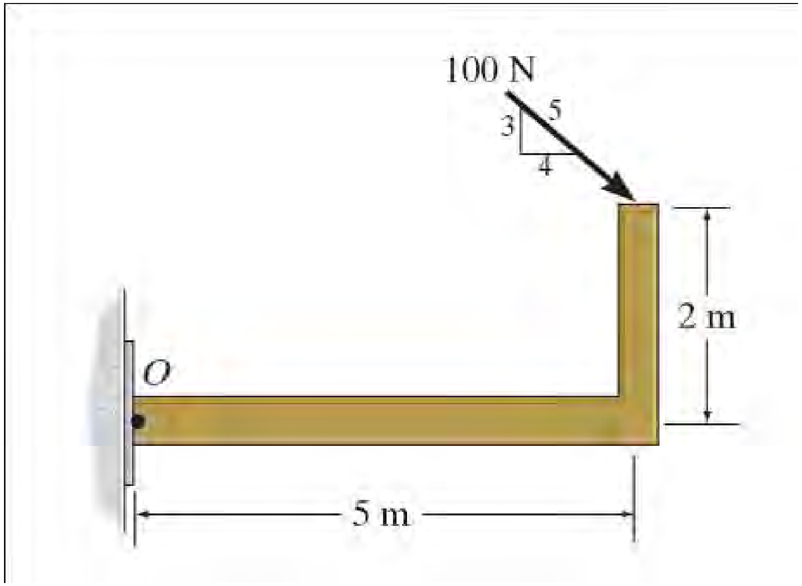
$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

By expanding the above equation using 2 × 2 determinants (see Section 4.2), we get (sample units are N - m or lb - ft)

$$\boldsymbol{M_O} = (r_y\ F_Z\ - r_Z\ F_y)\ \boldsymbol{i}\ -\ (r_x\ F_z\ -\ r_z\ F_x)\ \boldsymbol{j}\ +\ (r_x\ F_y\ - r_y\ F_x)\ \boldsymbol{k}$$

The physical meaning of the above equation becomes evident by considering the force components separately and using a 2-D formulation.

EXAMPLE I



Given: A 100 N force is applied to the frame.

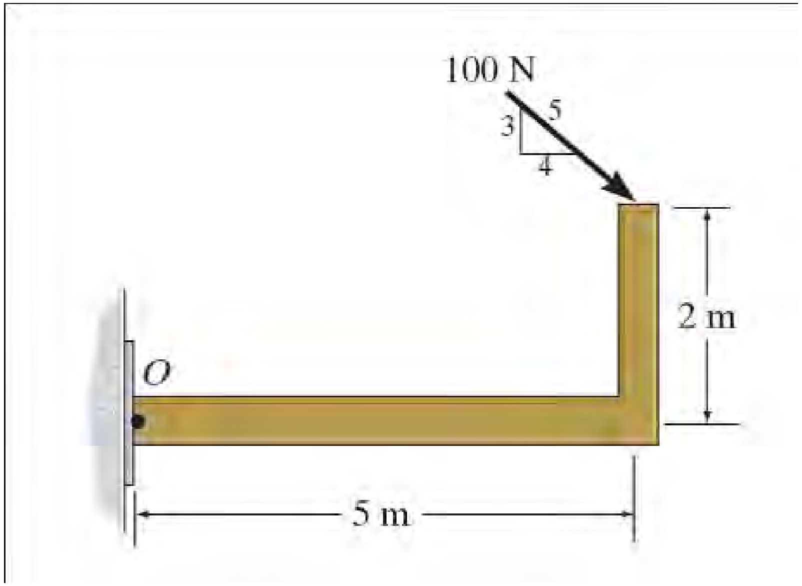
Find: The moment of the force at point O.

Plan:

1) Resolve the 100 N force along x and y-axes.

2) Determine M_O using a scalar analysis for the two force components and then add those two moments together..

EXAMPLE I (continued)



Solution

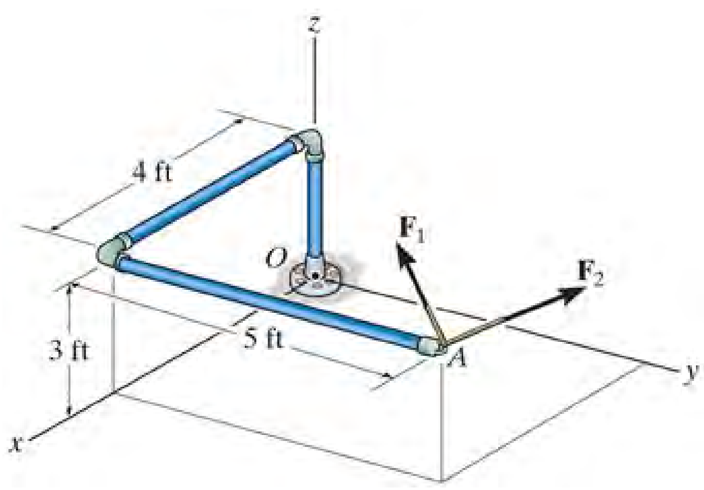
$+\uparrow F_y = -100\,(3/5)\text{ N}$

$+\rightarrow F_x = 100\,(4/5)\text{ N}$

$+\circlearrowleft M_O = \{-100\,(3/5)\text{N}\,(5\text{ m}) - (100)(4/5)\text{N}\,(2\text{ m})\}\ \text{N}\cdot\text{m}$

$= -460\ \text{N}\cdot\text{m}\circlearrowleft$ or $460\ \text{N}\cdot\text{m}$ CW

EXAMPLE II



Given: $F_1 = \{100\ i - 120\ j + 75\ k\}$lb

$F_2 = \{-200\ i + 250\ j + 100\ k\}$lb

Find: Resultant moment by the forces about point O.

Plan:

1) Find $F = F_1 + F_2$ and r_{OA}.

2) Determine $M_O = r_{OA} \times F$.

EXAMPLE II (continued)

Solution:

First, find the resultant force vector $\boldsymbol{F}$

$$\boldsymbol{F} = \boldsymbol{F_1} + \boldsymbol{F_2}$$

$$= \{ (100 - 200)\, \boldsymbol{i} + (-120 + 250)\, \boldsymbol{j} + (75 + 100)\, \boldsymbol{k} \} \text{ lb}$$

$$= \{-100\, \boldsymbol{i} + 130\, \boldsymbol{j} + 175\, \boldsymbol{k}\} \text{ lb}$$

Find the position vector $\boldsymbol{r_{OA}}$

$$\boldsymbol{r_{OA}} = \{4\, \boldsymbol{i} + 5\, \boldsymbol{j} + 3\, \boldsymbol{k}\} \text{ ft}$$

Then find the moment by using the vector cross product.

$$\boldsymbol{M_O} = \begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 4 & 5 & 3 \\ -100 & 130 & 175 \end{pmatrix} = [\{5(175) - 3(130)\}\, \boldsymbol{i} - \{4(175) - 3(-100)\}\, \boldsymbol{j} + \{4(130) - 5(-100)\}\, \boldsymbol{k}] \text{ ft·lb}$$

$$= \{485\, \boldsymbol{i} - 1000\, \boldsymbol{j} + 1020\, \boldsymbol{k}\} \text{ ft·lb}$$

READING QUIZ

1. What is the moment of the 12 N force about point A (M_A)?

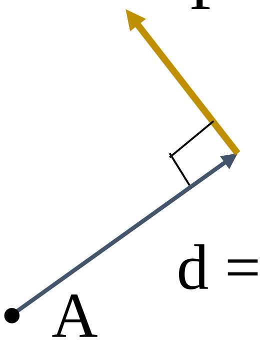
A) 3 N·m

B) 36 N·m

C) 12 N·m

D) (12/3) N·m

E) 7 N·m



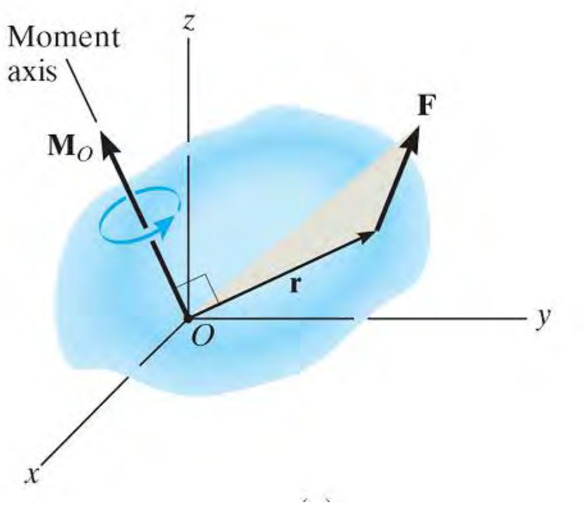
2. The moment of force ***F*** about point O is defined as $\boldsymbol{M_O}$ = ___________ .

A) ***r* x *F***

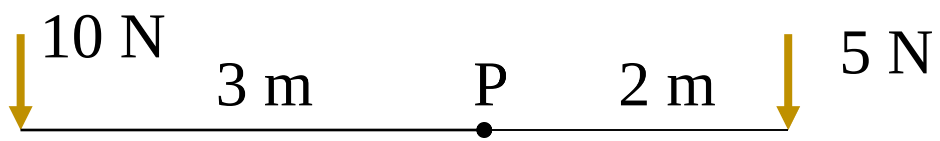
B) ***F* x *r***

C) ***r* • *F***

D) ***r* * *F***



ATTENTION QUIZ



1. Using the CCW direction as positive, the net moment of the two forces about point P is

A) 10 N ·m B) 20 N ·m C) - 20 N ·m

D) 40 N ·m E) - 40 N ·m

2. If ***r*** = { 5 ***j*** } m and ***F*** = { 10 ***k*** } N, the moment

r* x *F equals { ________ } N·m.

A) 50 ***i*** B) 50 ***j*** C) –50 ***i***

D) – 50 ***j*** E) 0

CONCEPT QUIZ

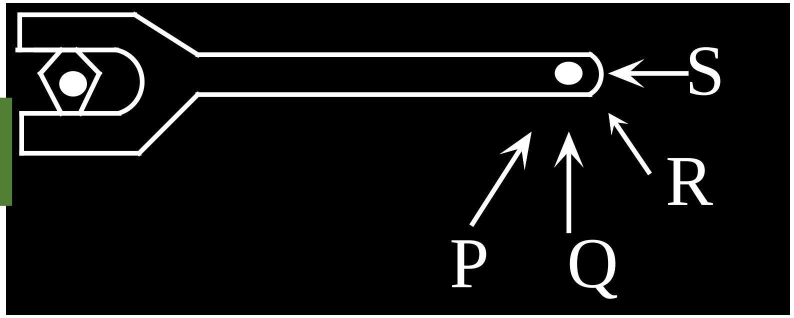
1. If a force of magnitude F can be applied in four different 2-D configurations (P,Q,R, & S), select the cases resulting in the maximum and minimum torque values on the nut. (Max, Min).

A) (Q, P)

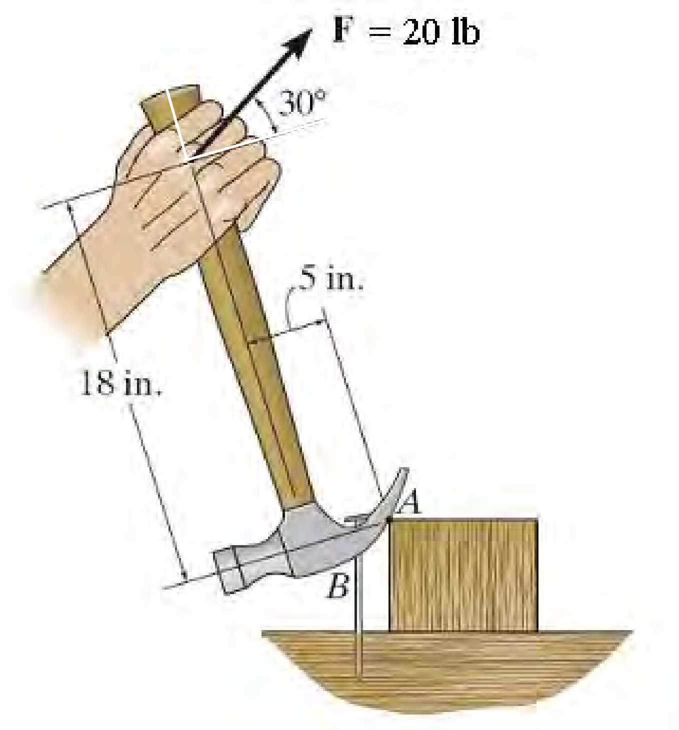
B) (R, S)

C) (P, R)

D) (Q, S)



GROUP PROBLEM SOLVING I

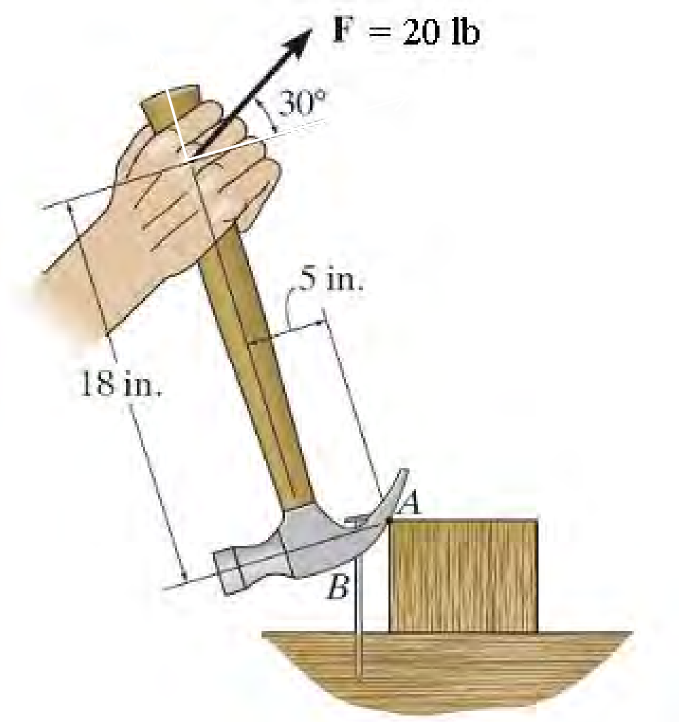


Given: A 20 lb force is applied to the hammer.

Find: The moment of the force at A.

Plan:

GROUP PROBLEM SOLVING I



Given: A 20 lb force is applied to the hammer.

Find: The moment of the force at A.

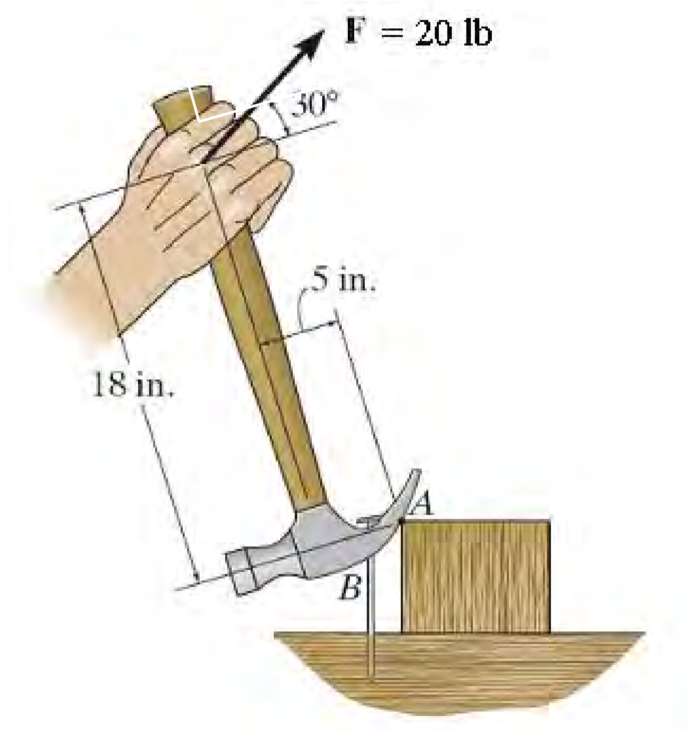
Plan:

Since this is a 2-D problem:

1) Resolve the 20 lb force along the handle's x and y axes.

2) Determine M_A using a scalar analysis.

GROUP PROBLEM SOLVING I (continued)



Solution:

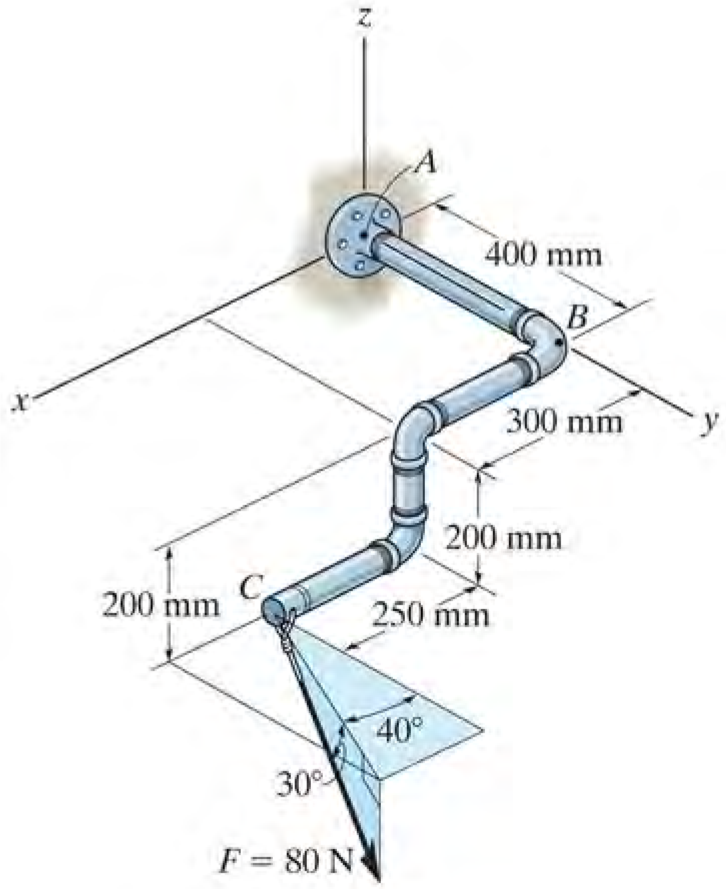
$+\uparrow F_y = 20 \sin 30° \text{ lb}$

$+\rightarrow F_x = 20 \cos 30° \text{ lb}$

$$+\circlearrowleft M_A = \{-(20 \cos 30°)\text{lb}\ (18 \text{ in}) - (20 \sin 30°)\text{lb}\ (5 \text{ in})\}$$

$$= -\,361.77 \text{ lb}\cdot\text{in} = 362 \text{ lb}\cdot\text{in (clockwise or CW)}$$

GROUP PROBLEM SOLVING II

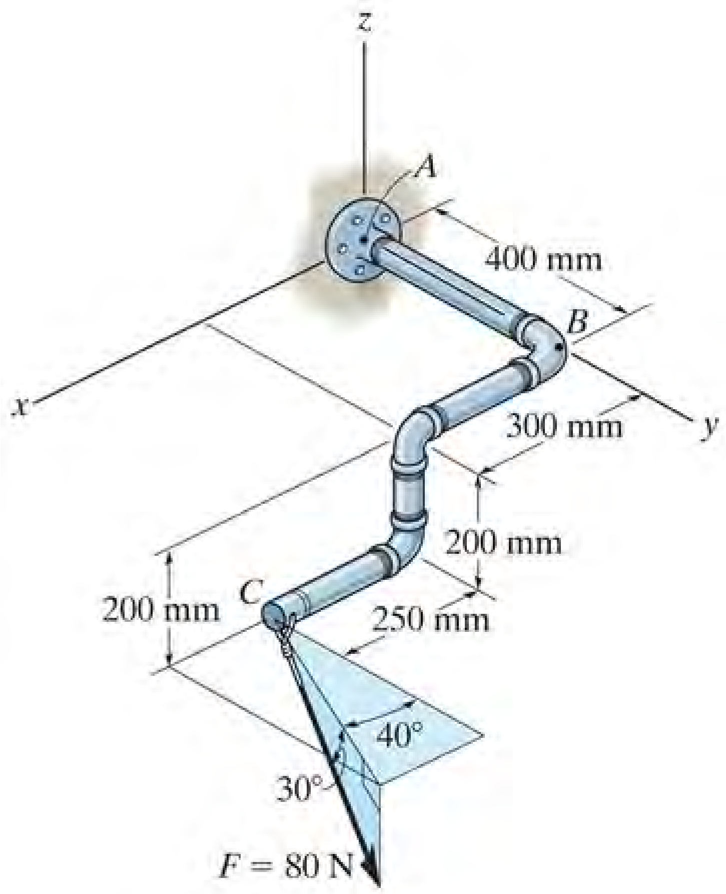


Given: The force and geometry shown.

Find: Moment of F about point A

Plan:

GROUP PROBLEM SOLVING II



Given: The force and geometry shown.

Find: Moment of F about point A

Plan:

1) Find $\boldsymbol{F}$ and $\boldsymbol{r}_{AC}$.

2) Determine $\boldsymbol{M}_A = \boldsymbol{r}_{AC} \times \boldsymbol{F}$

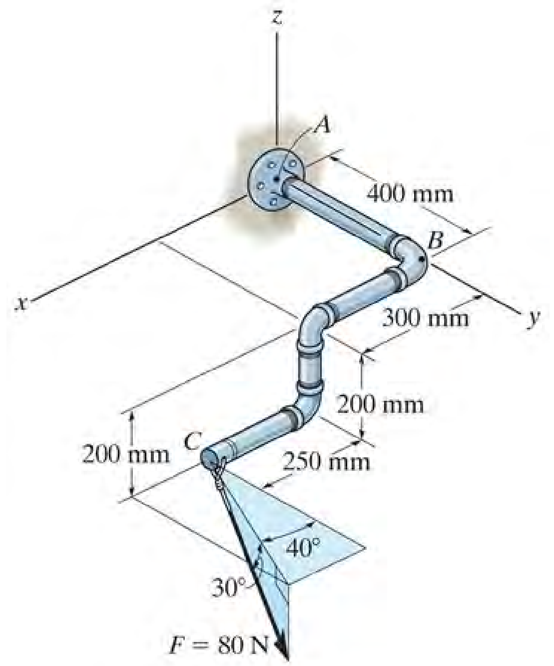
GROUP PROBLEM SOLVING II (continued)

Solution:

$\boldsymbol{F} = \{ (80 \cos 30) \sin 40\ \boldsymbol{i}$

$+ (80 \cos 30) \cos 40\ \boldsymbol{j} - 80 \sin 30\ \boldsymbol{k} \}$ N

$= \{44.53\ \boldsymbol{i} + 53.07\ \boldsymbol{j} - 40\ \boldsymbol{k}\}$ N



$\boldsymbol{r}_{AC} = \{0.55\ \boldsymbol{i} + 0.4\ \boldsymbol{j} - 0.2\ \boldsymbol{k}\}$ m

Find the moment by using the cross product.

$$\boldsymbol{M}_A = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0.55 & 0.4 & -0.2 \\ 44.53 & 53.07 & -40 \end{vmatrix}$$

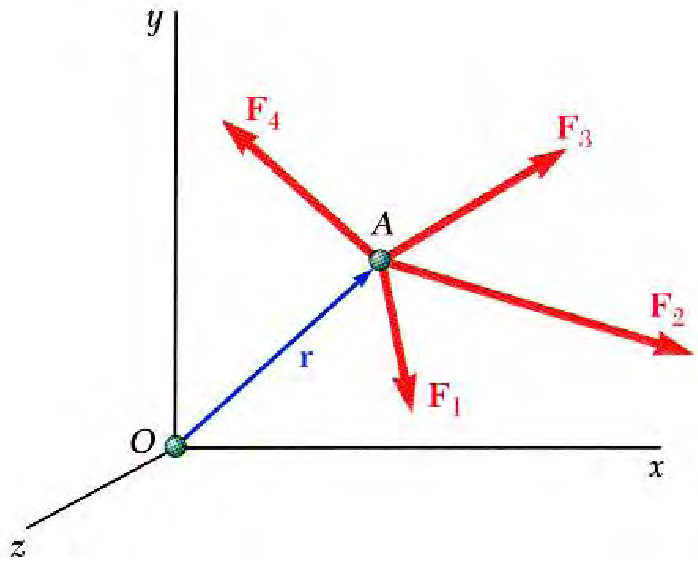
$= \{ -5.39\ \boldsymbol{i} + 13.1\ \boldsymbol{j} + 11.4\ \boldsymbol{k} \}$ N·m

Varignon's Theorem

- The moment about a give point *O* of the resultant of several concurrent forces is equal to the sum of the moments of the various moments about the same point *O*.

$$\vec{r} \times \left(\vec{F}_1 + \vec{F}_2 + \cdots\right) = \vec{r} \times \vec{F}_1 + \vec{r} \times \vec{F}_2 + \cdots$$

- Varigon's Theorem makes it possible to replace the direct determination of the moment of a force ***F*** by the moments of two or more component forces of ***F***.

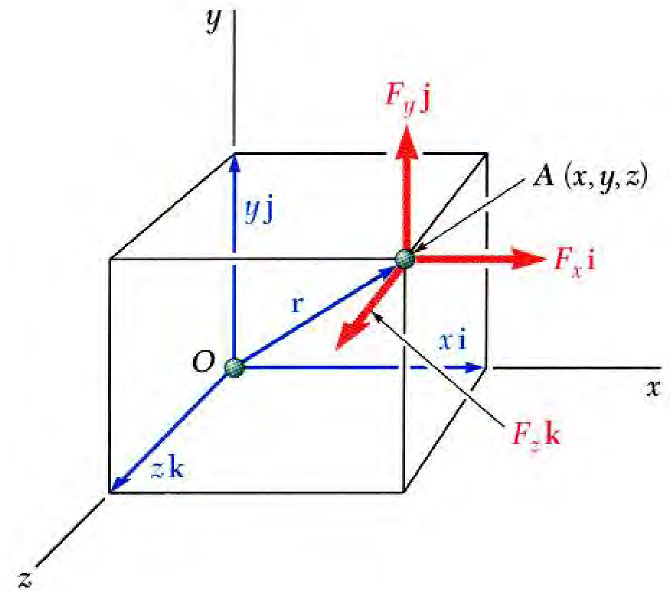


Rectangular Components of the Moment of a Force

The moment of ***F*** about *O*,

$$\vec{M}_O = \vec{r} \times \vec{F}, \qquad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$$



$$\vec{M}_O = M_x\vec{i} + M_y\vec{j} + M_z\vec{k}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

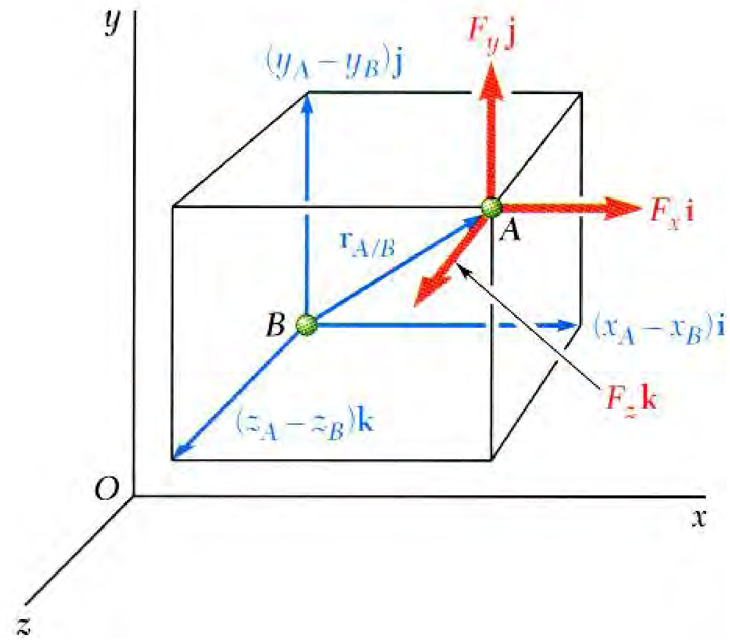
$$= \left(yF_z - zF_y\right)\vec{i} + \left(zF_x - xF_z\right)\vec{j} + \left(xF_y - yF_x\right)\vec{k}$$

Rectangular Components of the Moment of a Force

The moment of ***F*** about *B*,

$$\vec{M}_B = \vec{r}_{A/B} \times \vec{F}$$

$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B$$
$$= (x_A - x_B)\vec{i} + (y_A - y_B)\vec{j} + (z_A - z_B)\vec{k}$$
$$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$$



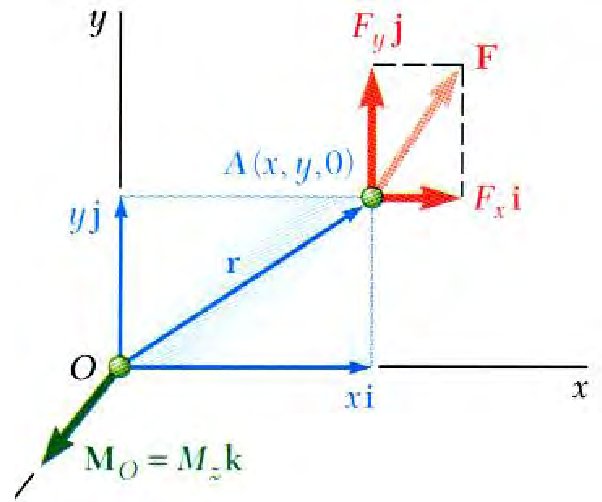
$$\vec{M}_B = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ (x_A - x_B) & (y_A - y_B) & (z_A - z_B) \\ F_x & F_y & F_z \end{vmatrix}$$

Rectangular Components of the Moment of a Force

For two-dimensional structures,

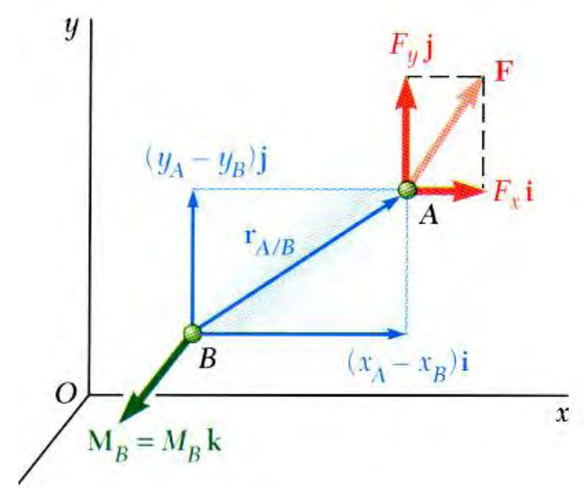
$$\vec{M}_O = \left(xF_y - yF_z\right)\vec{k}$$

$$M_O = M_Z$$
$$= xF_y - yF_z$$

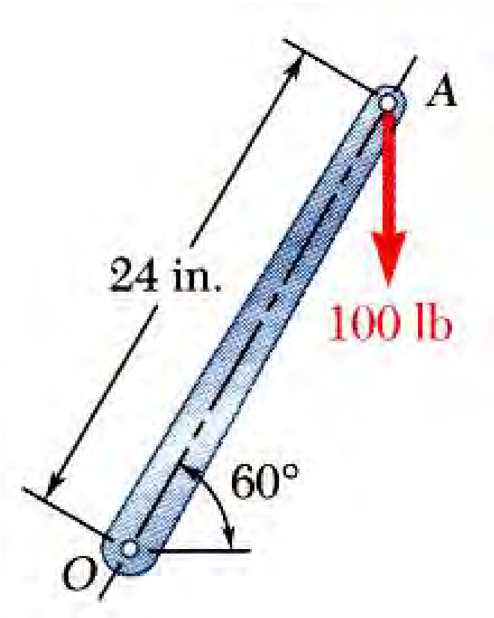


$$\vec{M}_O = \left[(x_A - x_B)F_y - (y_A - y_B)F_z\right]\vec{k}$$

$$M_O = M_Z$$
$$= (x_A - x_B)F_y - (y_A - y_B)F_z$$



Sample Problem



A 100-lb vertical force is applied to the end of a lever which is attached to a shaft at *O*.

Determine:

a) moment about *O*,

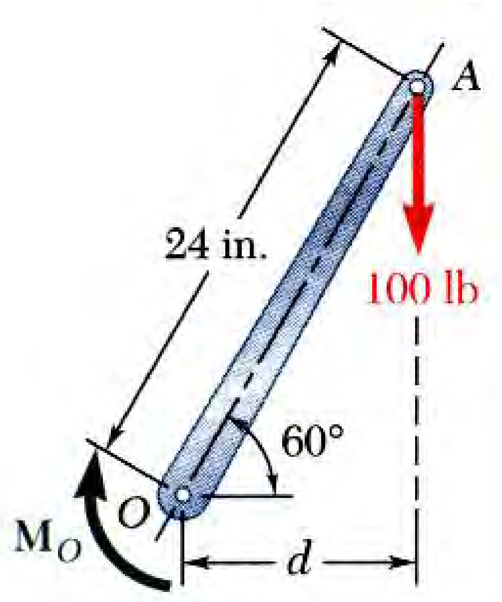
b) horizontal force at *A* which creates the same moment,

c) smallest force at A which produces the same moment,

d) location for a 240-lb vertical force to produce the same moment,

e) whether any of the forces from b, c, and d is equivalent to the original force.

Sample Problem



a) Moment about O is equal to the product of the force and the perpendicular distance between the line of action of the force and O. Since the force tends to rotate the lever clockwise, the moment vector is into the plane of the paper.

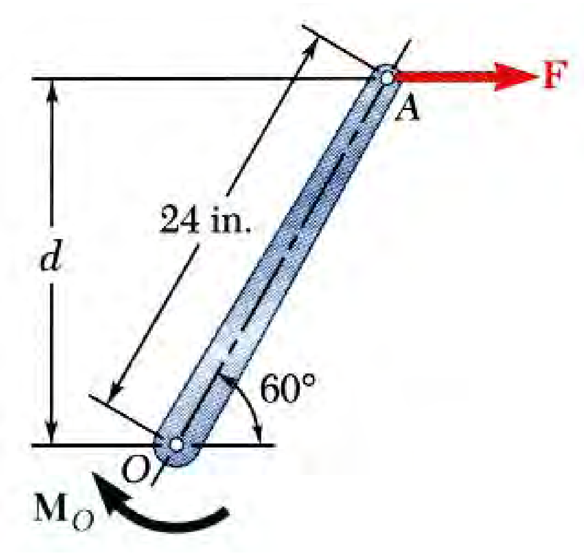
$$M_O = Fd$$

$$d = (24\,\text{in.})\cos 60° = 12\,\text{in.}$$

$$M_O = (100\,\text{lb})(12\,\text{in.})$$

$$\boxed{M_O = 1200\,\text{lb}\cdot\text{in}}$$

Sample Problem



c) Horizontal force at *A* that produces the same moment,

$$d = (24\,\text{in.})\sin 60° = 20.8\,\text{in.}$$

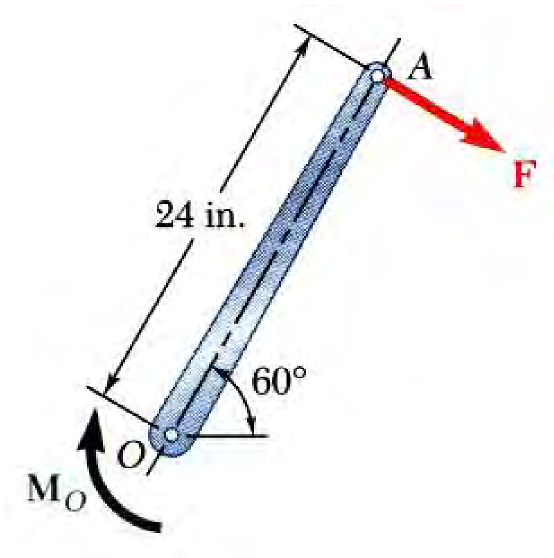
$$M_O = Fd$$

$$1200\,\text{lb}\cdot\text{in.} = F(20.8\,\text{in.})$$

$$F = \frac{1200\,\text{lb}\cdot\text{in.}}{20.8\,\text{in.}}$$

$$\boxed{F = 57.7\,\text{lb}}$$

Sample Problem



c) The smallest force A to produce the same moment occurs when the perpendicular distance is a maximum or when F is perpendicular to OA.

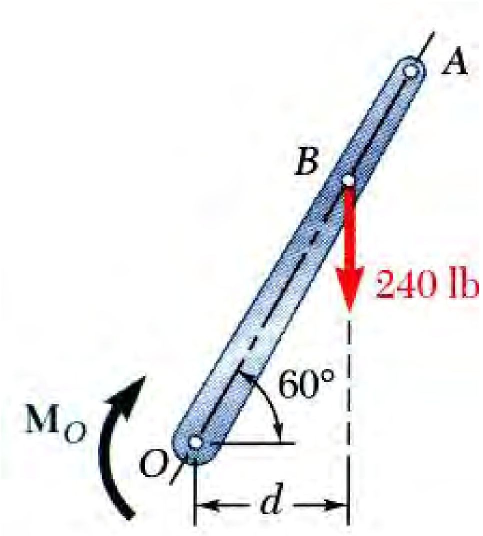
$$M_O = Fd$$

$$1200\,\text{lb}\cdot\text{in.} = F(24\,\text{in.})$$

$$F = \frac{1200\,\text{lb}\cdot\text{in.}}{24\,\text{in.}}$$

$$\boxed{F = 50\,\text{lb}}$$

Sample Problem



d) To determine the point of application of a 240 lb force to produce the same moment,

$$M_O = Fd$$

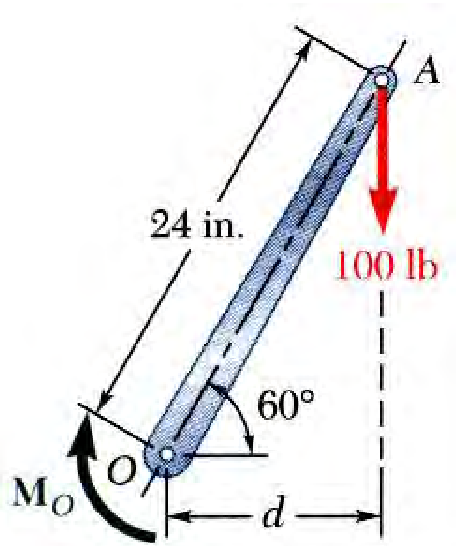
$$1200\ \text{lb}\cdot\text{in.} = (240\ \text{lb})d$$

$$d = \frac{1200\ \text{lb}\cdot\text{in.}}{240\ \text{lb}} = 5\ \text{in.}$$

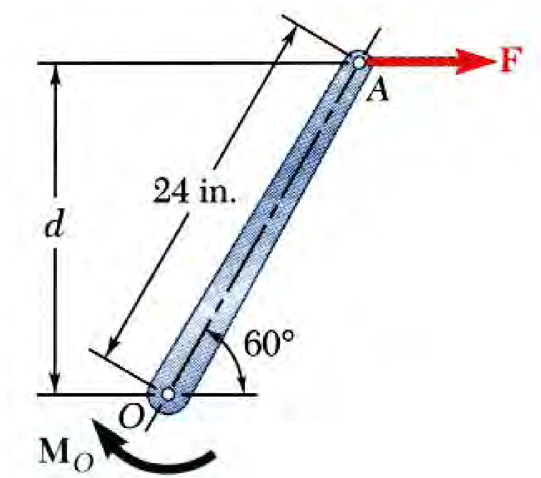
$$OB\cos 60° = 5\ \text{in.}$$

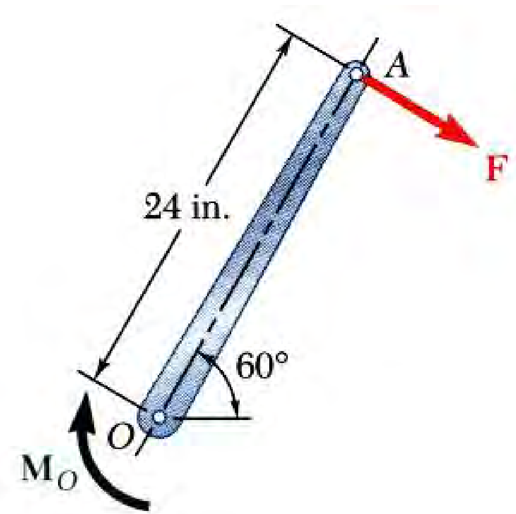
$$\boxed{OB = 10\ \text{in.}}$$

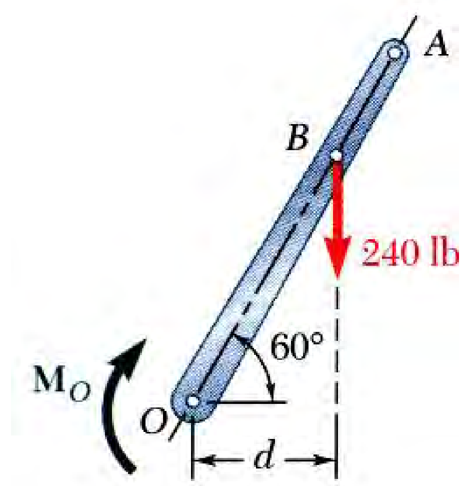
Sample Problem



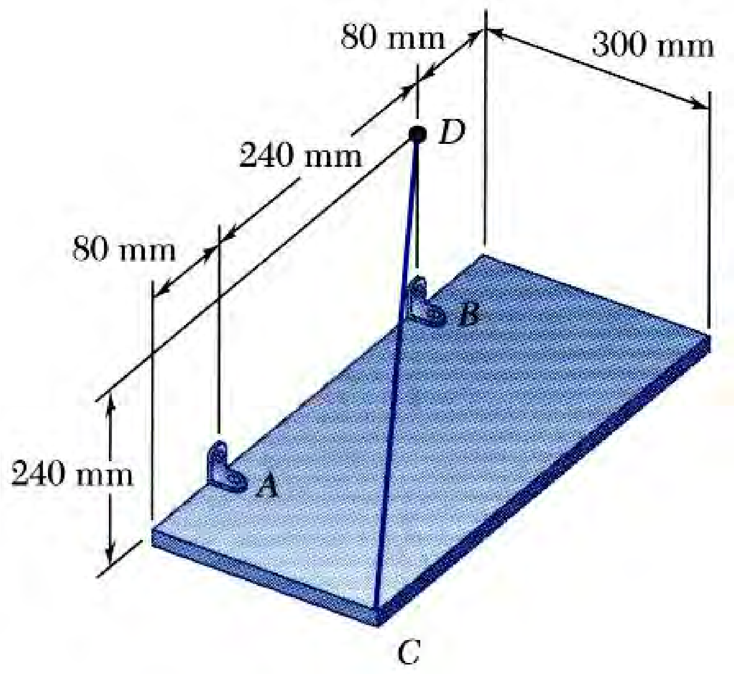
e) Although each of the forces in parts b), c), and d) produces the same moment as the 100 lb force, none are of the same magnitude and sense, or on the same line of action. None of the forces is equivalent to the 100 lb force.







Sample Problem



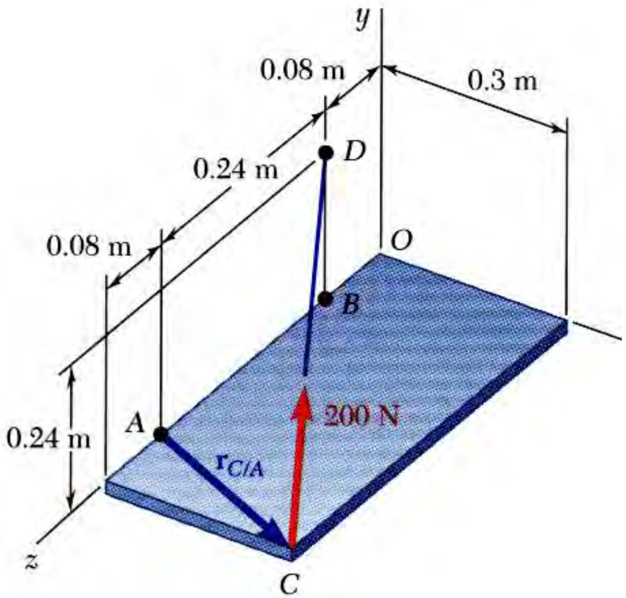
The rectangular plate is supported by the brackets at *A* and *B* and by a wire *CD*. Knowing that the tension in the wire is 200 N, determine the moment about *A* of the force exerted by the wire at *C*.

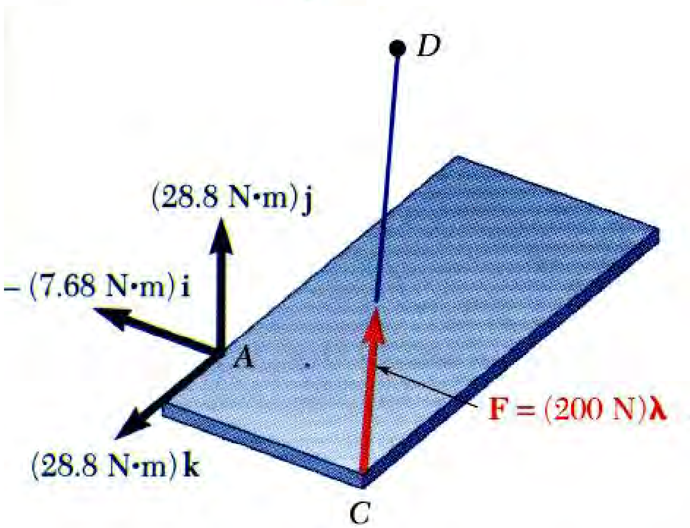
SOLUTION:

The moment M_A of the force *F* exerted by the wire is obtained by evaluating the vector product,

$$\vec{M}_A = \vec{r}_{C/A} \times \vec{F}$$

Sample Problem





SOLUTION:

$$\vec{M}_A = \vec{r}_{C/A} \times \vec{F}$$

$$\vec{r}_{C/A} = \vec{r}_C - \vec{r}_A = (0.3\,\text{m})\vec{i} + (0.08\,\text{m})\vec{j}$$

$$\vec{F} = F\vec{\lambda} = (200\ \text{N})\frac{\vec{r}_{C/D}}{r_{C/D}}$$

$$= (200\ \text{N})\frac{-(0.3\,\text{m})\vec{i} + (0.24\,\text{m})\vec{j} - (0.32\,\text{m})\vec{k}}{0.5\,\text{m}}$$

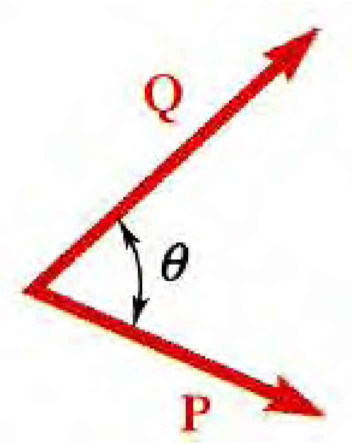
$$= -(120\ \text{N})\vec{i} + (96\ \text{N})\vec{j} - (128\ \text{N})\vec{k}$$

$$\vec{M}_A = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.3 & 0 & 0.08 \\ -120 & 96 & -128 \end{vmatrix}$$

$$\boxed{\vec{M}_A = -(7.68\,\text{N}\cdot\text{m})\vec{i} + (28.8\,\text{N}\cdot\text{m})\vec{j} + (28.8\,\text{N}\cdot\text{m})\vec{k}}$$

Scalar Product of Two Vectors

- The *scalar product* or *dot product* between two vectors ***P*** and ***Q*** is defined as
  $\vec{P} \bullet \vec{Q} = PQ\cos\theta \quad (\text{scalar result})$



- Scalar products:
  - are commutative, $\vec{P} \bullet \vec{Q} = \vec{Q} \bullet \vec{P}$
  - are distributive, $\vec{P} \bullet (\vec{Q}_1 + \vec{Q}_2) = \vec{P} \bullet \vec{Q}_1 + \vec{P} \bullet \vec{Q}_2$
  - are not associative, $(\vec{P} \bullet \vec{Q}) \bullet \vec{S} = \text{undefined}$

- Scalar products with Cartesian unit components,

$$\vec{P} \bullet \vec{Q} = (P_x\vec{i} + P_y\vec{j} + P_z\vec{k}) \bullet (Q_x\vec{i} + Q_y\vec{j} + Q_z\vec{k})$$

$$\vec{i} \bullet \vec{i} = 1 \quad \vec{j} \bullet \vec{j} = 1 \quad \vec{k} \bullet \vec{k} = 1 \quad \vec{i} \bullet \vec{j} = 0 \quad \vec{j} \bullet \vec{k} = 0 \quad \vec{k} \bullet \vec{i} = 0$$

$$\vec{P} \bullet \vec{Q} = P_xQ_x + P_yQ_y + P_zQ_z$$

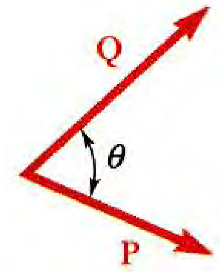
$$\vec{P} \bullet \vec{P} = P_x^2 + P_y^2 + P_z^2 = P^2$$

Scalar Product of Two Vectors: Applications

- Angle between two vectors:

$$\vec{P} \bullet \vec{Q} = PQ\cos\theta = P_x Q_x + P_y Q_y + P_z Q_z$$

$$\cos\theta = \frac{P_x Q_x + P_y Q_y + P_z Q_z}{PQ}$$

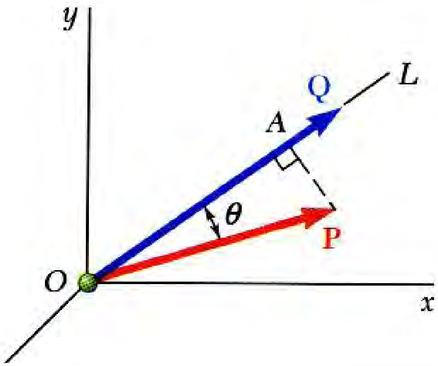


- Projection of a vector on a given axis:

$$P_{OL} = P\cos\theta = \text{projection of } P \text{ along } OL$$

$$\vec{P} \bullet \vec{Q} = PQ\cos\theta$$

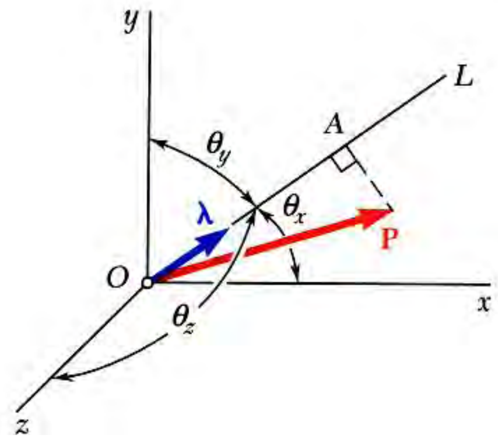
$$\frac{\vec{P} \bullet \vec{Q}}{Q} = P\cos\theta = P_{OL}$$



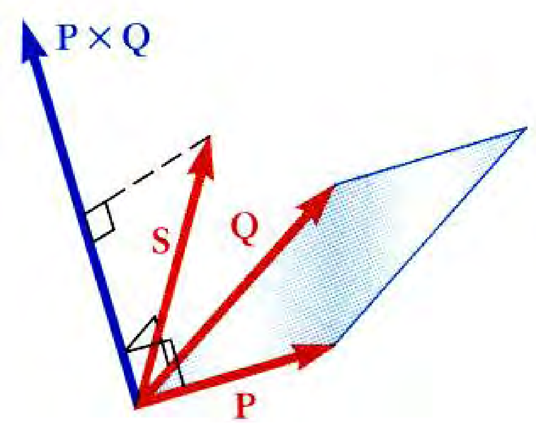
- For an axis defined by a unit vector:

$$P_{OL} = \vec{P} \bullet \vec{\lambda}$$

$$= P_x \cos\theta_x + P_y \cos\theta_y + P_z \cos\theta_z$$



Mixed Triple Product of Three Vectors



- Mixed triple product of three vectors,

  $\vec{S} \bullet \left(\vec{P} \times \vec{Q}\right) =$ scalar result

- The six mixed triple products formed from ***S***, ***P***, and ***Q*** have equal magnitudes but not the same sign,

$$\vec{S} \bullet \left(\vec{P} \times \vec{Q}\right) = \vec{P} \bullet \left(\vec{Q} \times \vec{S}\right) = \vec{Q} \bullet \left(\vec{S} \times \vec{P}\right)$$

$$= -\vec{S} \bullet \left(\vec{Q} \times P\right) = -\vec{P} \bullet \left(\vec{S} \times \vec{Q}\right) = -\vec{Q} \bullet \left(\vec{P} \times \vec{S}\right)$$

- Evaluating the mixed triple product,

$$\vec{S} \bullet \left(\vec{P} \times \vec{Q}\right) = S_x\left(P_y Q_z - P_z Q_y\right) + S_y\left(P_z Q_x - P_x Q_z\right) + S_z\left(P_x Q_y - P_y Q_x\right)$$

$$= \begin{vmatrix} S_x & S_y & S_z \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix}$$

Moment of a Force About a Given Axis

- Moment $\boldsymbol{M_O}$ of a force $\boldsymbol{F}$ applied at the point $\boldsymbol{A}$ about a point $\boldsymbol{O}$,

$$\vec{M}_O = \vec{r} \times \vec{F}$$

- Scalar moment M_{OL} about an axis $\boldsymbol{OL}$ is the projection of the moment vector $\boldsymbol{M_O}$ onto the axis,

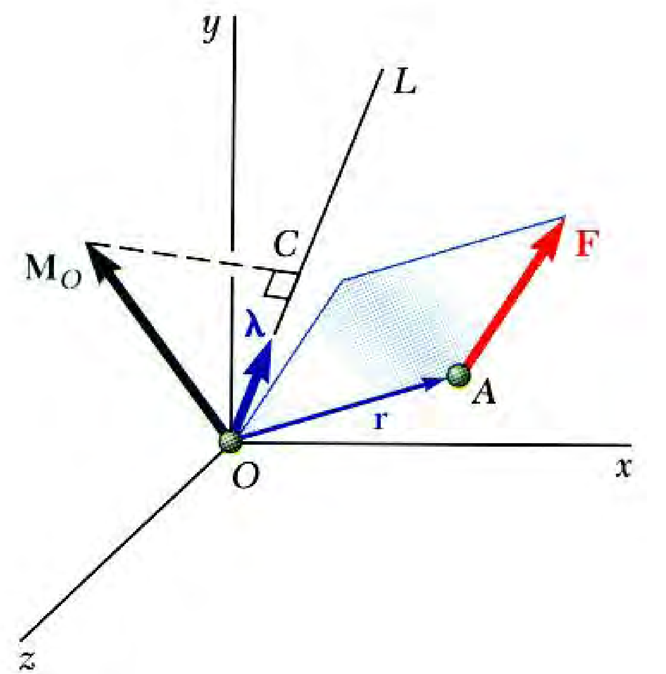
$$M_{OL} = \vec{\lambda} \bullet \vec{M}_O = \vec{\lambda} \bullet \left(\vec{r} \times \vec{F} \right)$$

- Moments of $\boldsymbol{F}$ about the coordinate axes,

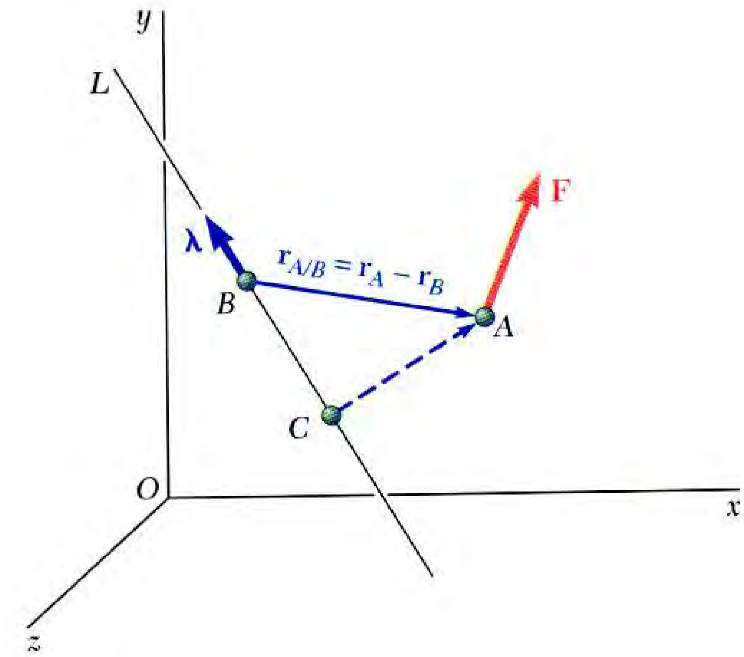
$$M_x = yF_z - zF_y$$

$$M_y = zF_x - xF_z$$

$$M_z = xF_y - yF_x$$



Moment of a Force About a Given Axis

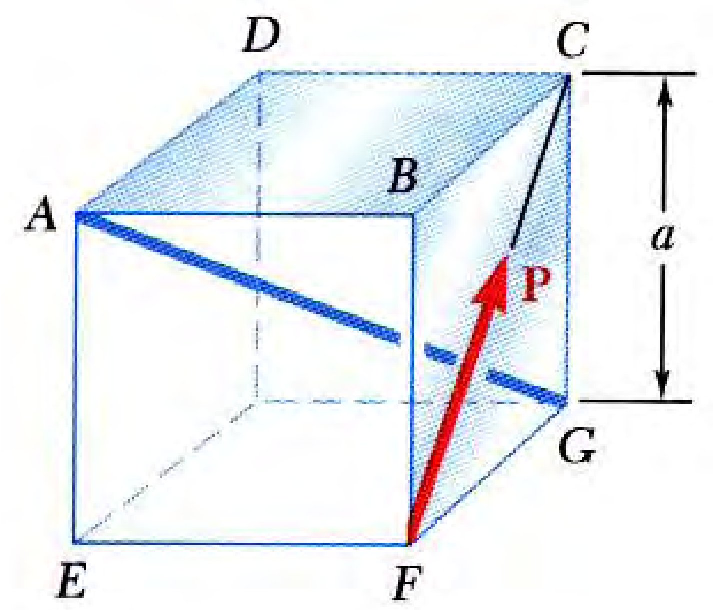


- Moment of a force about an arbitrary axis,

$$
\begin{aligned}
M_{BL} &= \vec{\lambda} \bullet \vec{M}_B \\
&= \vec{\lambda} \bullet \left(\vec{r}_{A/B} \times \vec{F} \right) \\
\vec{r}_{A/B} &= \vec{r}_A - \vec{r}_B
\end{aligned}
$$

- The result is independent of the point *B* along the given axis.

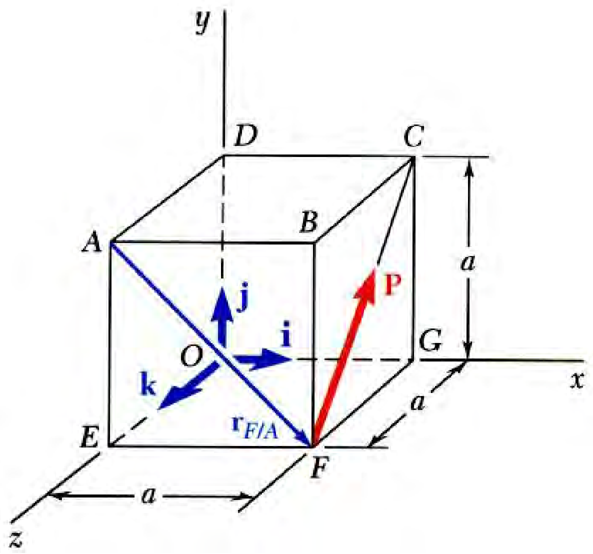
Sample Problem



A cube is acted on by a force ***P*** as shown. Determine the moment of ***P***

a) about *A*
b) about the edge *AB* and
c) about the diagonal *AG* of the cube.
d) Determine the perpendicular distance between *AG* and *FC*.

Sample Problem



- Moment of ***P*** about *A*,

$$\vec{M}_A = \vec{r}_{F/A} \times \vec{P}$$

$$\vec{r}_{F/A} = a\vec{i} - a\vec{j} = a\left(\vec{i} - \vec{j}\right)$$

$$\vec{P} = P\left(\sqrt{2}\,\vec{i} + \sqrt{2}\,\vec{j}\right) = P\sqrt{2}\left(\vec{i} + \vec{j}\right)$$

$$\vec{M}_A = a\left(\vec{i} - \vec{j}\right) \times P\sqrt{2}\left(\vec{i} + \vec{j}\right)$$

$$\boxed{\vec{M}_A = \left(aP\sqrt{2}\right)\left(\vec{i} + \vec{j} + \vec{k}\right)}$$

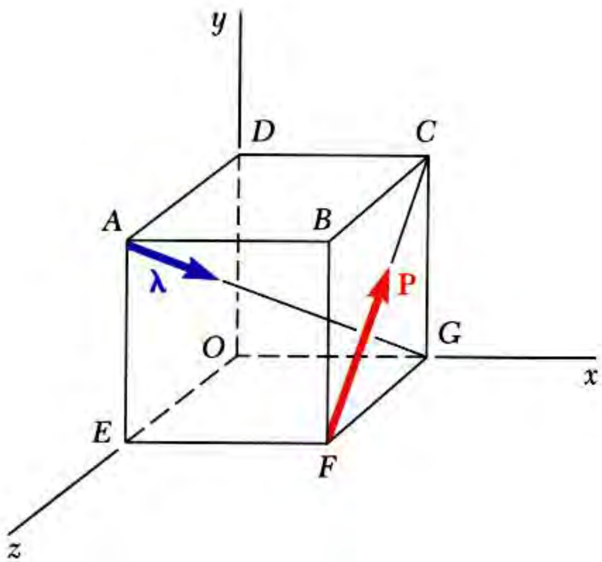
- Moment of ***P*** about *AB*,

$$M_{AB} = \vec{i} \bullet \vec{M}_A$$

$$= \vec{i} \bullet \left(aP\sqrt{2}\right)\left(\vec{i} + \vec{j} + \vec{k}\right)$$

$$\boxed{M_{AB} = aP\sqrt{2}}$$

Sample Problem



- Moment of ***P*** about the diagonal *AG*,

$$M_{AG} = \vec{\lambda} \bullet \vec{M}_A$$

$$\vec{\lambda} = \frac{\vec{r}_{A/G}}{r_{A/G}} = \frac{a\vec{i} - a\vec{j} - a\vec{k}}{a\sqrt{3}} = \frac{1}{\sqrt{3}}\left(\vec{i} - \vec{j} - \vec{k}\right)$$

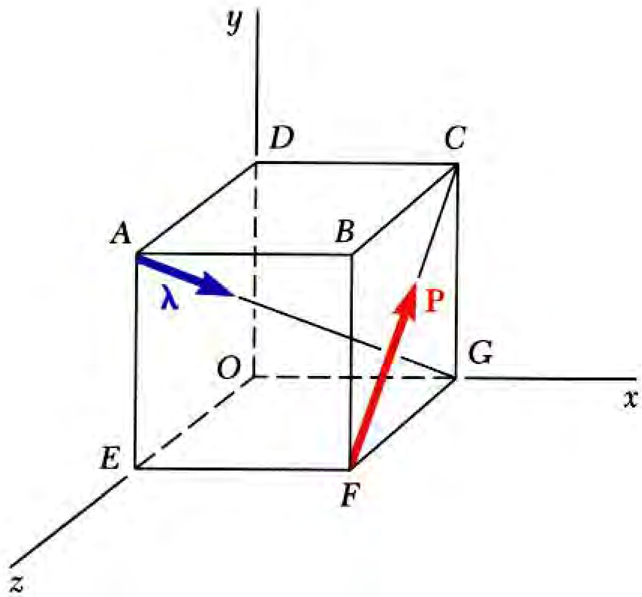
$$\vec{M}_A = \frac{aP}{\sqrt{2}}\left(\vec{i} + \vec{j} + \vec{k}\right)$$

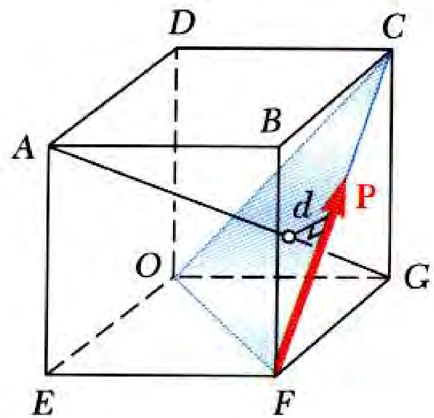
$$M_{AG} = \frac{1}{\sqrt{3}}\left(\vec{i} - \vec{j} - \vec{k}\right) \bullet \frac{aP}{\sqrt{2}}\left(\vec{i} + \vec{j} + \vec{k}\right)$$

$$= \frac{aP}{\sqrt{6}}(1 - 1 - 1)$$

$$\boxed{M_{AG} = -\frac{aP}{\sqrt{6}}}$$

Sample Problem





- Perpendicular distance between *AG* and *FC*,

$$\vec{P} \bullet \vec{\lambda} = \frac{P}{\sqrt{2}}\left(\vec{j} - \vec{k}\right) \bullet \frac{1}{\sqrt{3}}\left(\vec{i} - \vec{j} - \vec{k}\right) = \frac{P}{\sqrt{6}}(0 - 1 + 1)$$
$$= 0$$

Therefore, *P* is perpendicular to *AG*.

$$\left|M_{AG}\right| = \frac{aP}{\sqrt{6}} = Pd$$

$$\boxed{d = \frac{a}{\sqrt{6}}}$$