STATICS

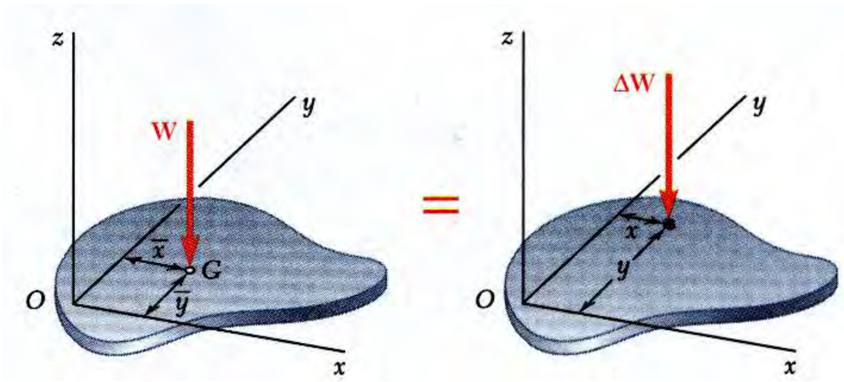
Distributed Forces:
Centroids and Centers of Gravity

Introduction

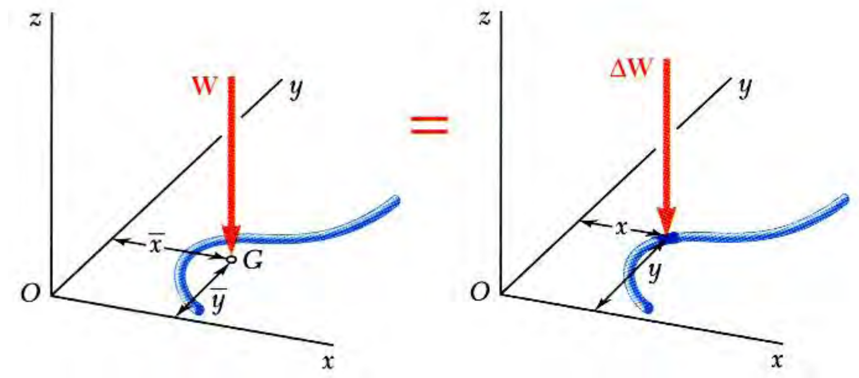
- The earth exerts a gravitational force on each of the particles forming a body. These forces can be replace by a single equivalent force equal to the weight of the body and applied at the *center of gravity* for the body.
- The *centroid of an area* is analogous to the center of gravity of a body. The concept of the *first moment of an area* is used to locate the centroid.
- Determination of the area of a *surface of revolution* and the volume of a *body of revolution* are accomplished with the *Theorems of Pappus-Guldinus*.

Center of Gravity of a 2D Body

- Center of gravity of a plate



- Center of gravity of a wire



$$\sum M_y \quad \bar{x}W = \sum x\Delta W$$

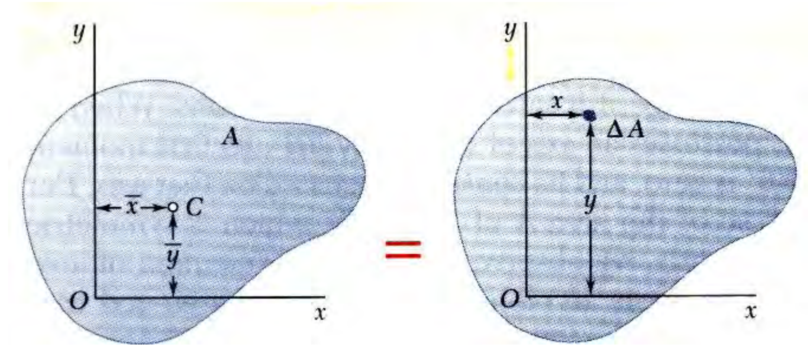
$$= \int x\, dW$$

$$\sum M_y \quad \bar{y}W = \sum y\Delta W$$

$$= \int y\, dW$$

Centroids and First Moments of Areas and Lines

- Centroid of an area



$$\bar{x}W = \int x\,dW$$

$$\bar{x}(\gamma A t) = \int x(\gamma t)\,dA$$

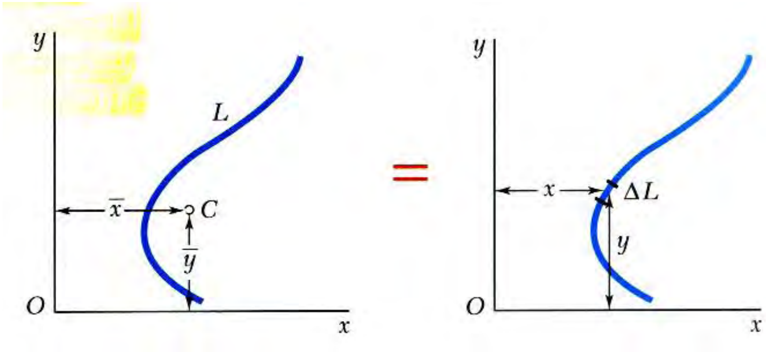
$$\bar{x}A = \int x\,dA = Q_y$$

$$= \text{first moment with respect to } y$$

$$\bar{y}A = \int y\,dA = Q_x$$

$$= \text{first moment with respect to } x$$

- Centroid of a line



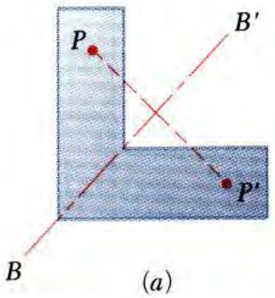
$$\bar{x}W = \int x\,dW$$

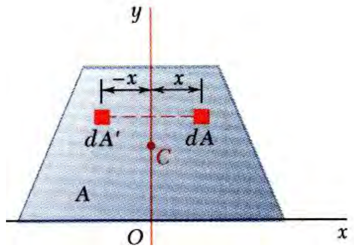
$$\bar{x}(\gamma L a) = \int x(\gamma a)\,dL$$

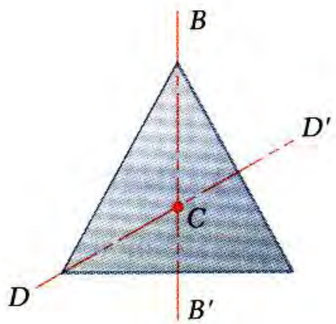
$$\bar{x}L = \int x\,dL$$

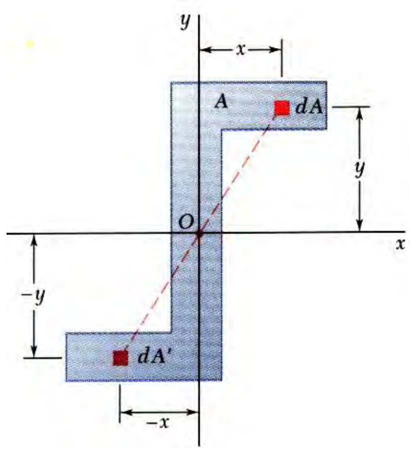
$$\bar{y}L = \int y\,dL$$

First Moments of Areas and Lines



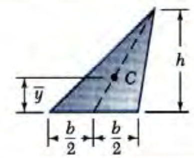
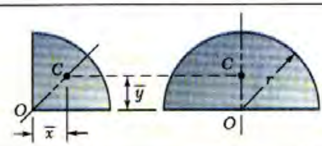
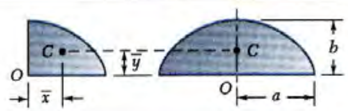
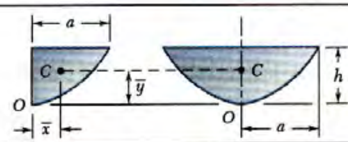
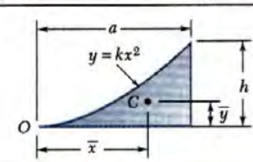
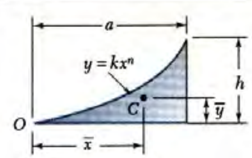
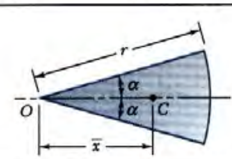




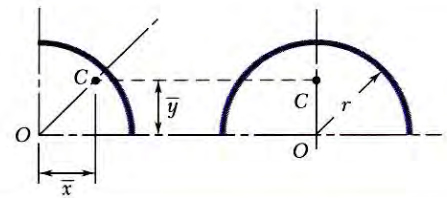
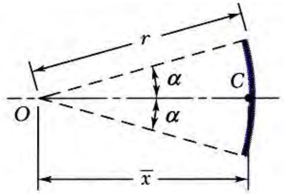


- An area is symmetric with respect to an axis *BB'* if for every point *P* there exists a point *P'* such that *PP'* is perpendicular to *BB'* and is divided into two equal parts by *BB'*.
- The first moment of an area with respect to a line of symmetry is zero.
- If an area possesses a line of symmetry, its centroid lies on that axis
- If an area possesses two lines of symmetry, its centroid lies at their intersection.
- An area is symmetric with respect to a center *O* if for every element *dA* at (*x*,*y*) there exists an area *dA'* of equal area at (-*x*,-*y*).
- The centroid of the area coincides with the center of symmetry.

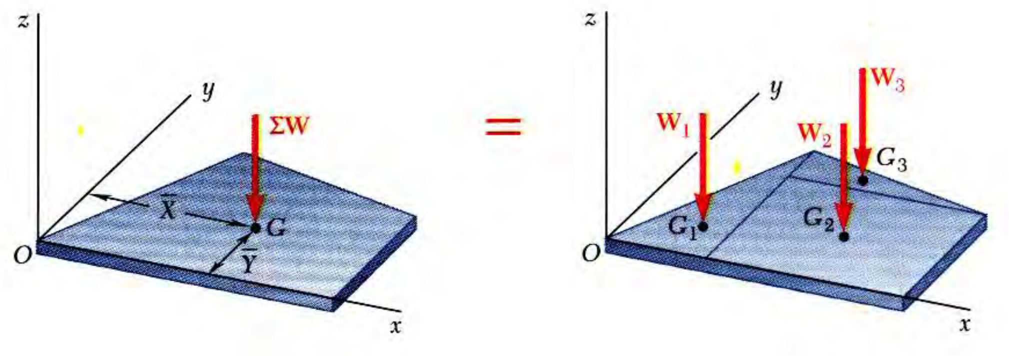
Centroids of Common Shapes of Areas

Shape		$\bar{x}$	$\bar{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
General spandrel		$\frac{n+1}{n+2}a$	$\frac{n+1}{4n+2}h$	$\frac{ah}{n+1}$
Circular sector		$\frac{2r\sin\alpha}{3\alpha}$	0	αr^2

Centroids of Common Shapes of Lines

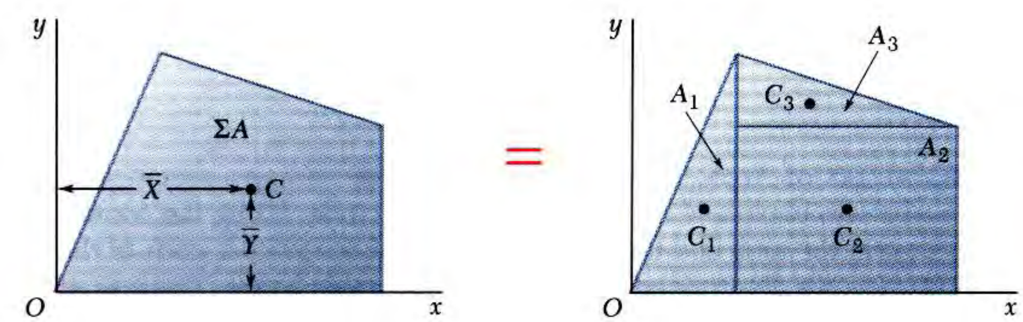
Shape		$\bar{x}$	$\bar{y}$	Length
Quarter-circular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semicircular arc		0	$\frac{2r}{\pi}$	πr
Arc of circle		$\frac{r \sin \alpha}{\alpha}$	0	$2\alpha r$

Composite Plates and Areas



- Composite plates

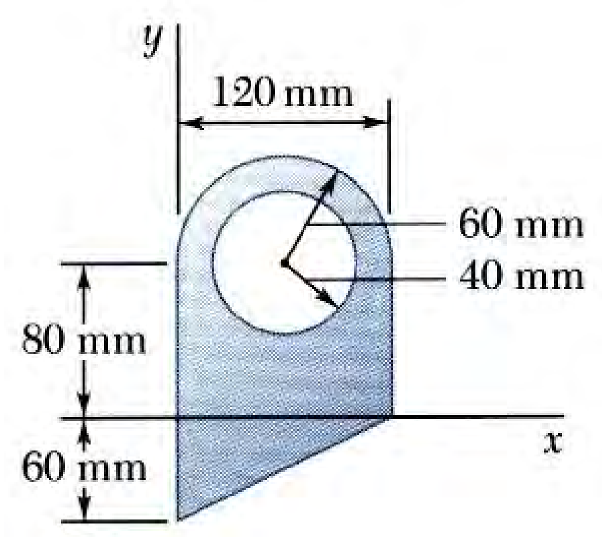
$$\bar{X}\sum W = \sum \bar{x} W$$
$$\bar{Y}\sum W = \sum \bar{y} W$$



- Composite area

$$\bar{X}\sum A = \sum \bar{x} A$$
$$\bar{Y}\sum A = \sum \bar{y} A$$

Sample Problem 5.1

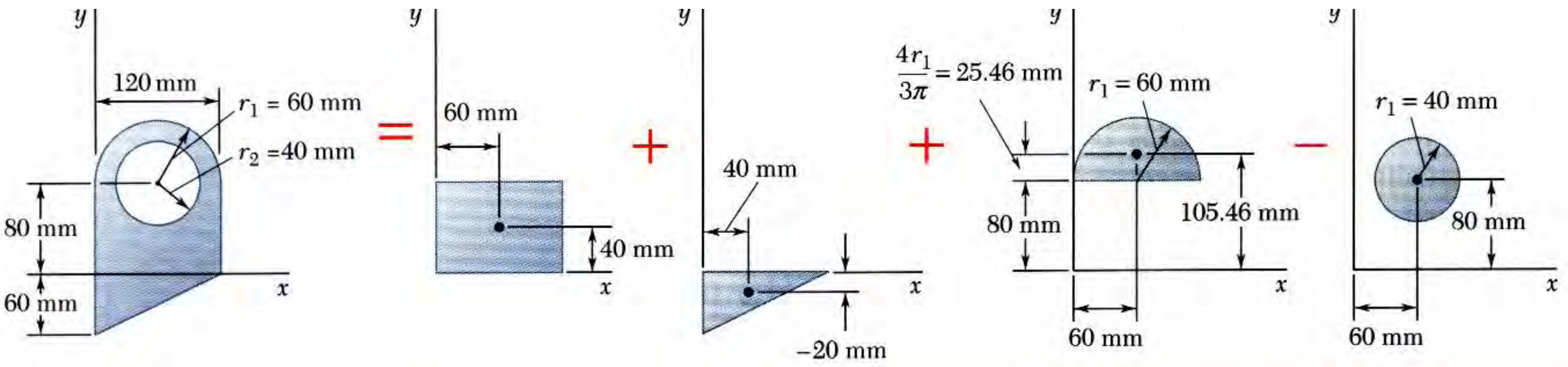


For the plane area shown, determine the first moments with respect to the *x* and *y* axes and the location of the centroid.

SOLUTION:

- Divide the area into a triangle, rectangle, and semicircle with a circular cutout.
- Calculate the first moments of each area with respect to the axes.
- Find the total area and first moments of the triangle, rectangle, and semicircle. Subtract the area and first moment of the circular cutout.
- Compute the coordinates of the area centroid by dividing the first moments by the total area.

Sample Problem 5.1



Component	A, mm²	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm³	$\bar{y}A$, mm³
Rectangle	$(120)(80) = 9.6 \times 10^3$	60	40	$+576 \times 10^3$	$+384 \times 10^3$
Triangle	$\frac{1}{2}(120)(60) = 3.6 \times 10^3$	40	−20	$+144 \times 10^3$	-72×10^3
Semicircle	$\frac{1}{2}\pi(60)^2 = 5.655 \times 10^3$	60	105.46	$+339.3 \times 10^3$	$+596.4 \times 10^3$
Circle	$-\pi(40)^2 = -5.027 \times 10^3$	60	80	-301.6×10^3	-402.2×10^3
	$\Sigma A = 13.828 \times 10^3$			$\Sigma\bar{x}A = +757.7 \times 10^3$	$\Sigma\bar{y}A = +506.2 \times 10^3$

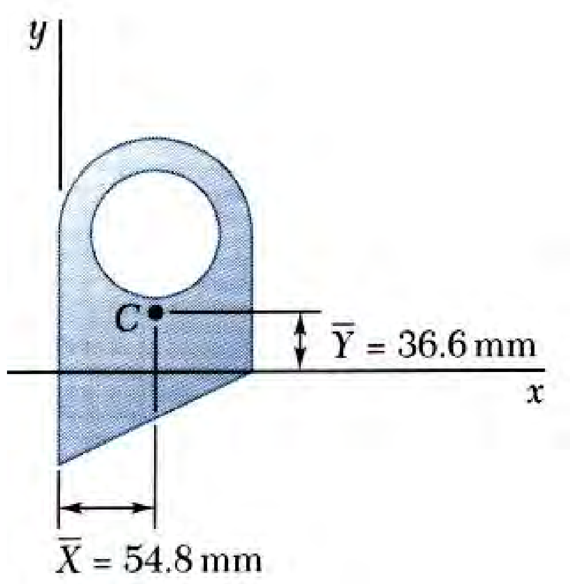
- Find the total area and first moments of the triangle, rectangle, and semicircle. Subtract the area and first moment of the circular cutout.

$$Q_x = +506.2 \times 10^3 \text{ mm}^3$$

$$Q_y = +757.7 \times 10^3 \text{ mm}^3$$

Sample Problem 5.1

- Compute the coordinates of the area centroid by dividing the first moments by the total area.



$$\bar{X} = \frac{\sum \bar{x}A}{\sum A} = \frac{+757.7 \times 10^3\,\text{mm}^3}{13.828 \times 10^3\,\text{mm}^2}$$

$$\bar{X} = 54.8\,\text{mm}$$

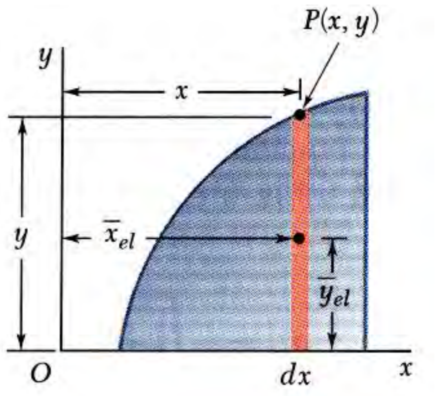
$$\bar{Y} = \frac{\sum \bar{y}A}{\sum A} = \frac{+506.2 \times 10^3\,\text{mm}^3}{13.828 \times 10^3\,\text{mm}^2}$$

$$\bar{Y} = 36.6\,\text{mm}$$

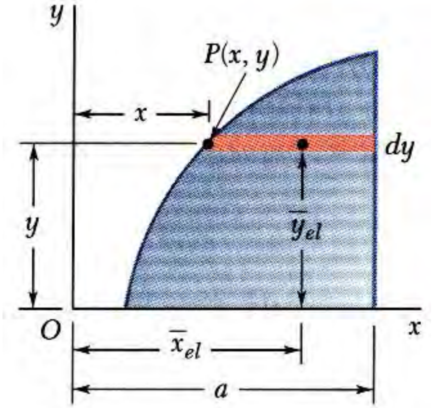
Determination of Centroids by Integration

$$\bar{x}A = \int x dA = \iint x\, dx dy = \int \bar{x}_{el}\, dA$$
$$\bar{y}A = \int y dA = \iint y\, dx dy = \int \bar{y}_{el}\, dA$$

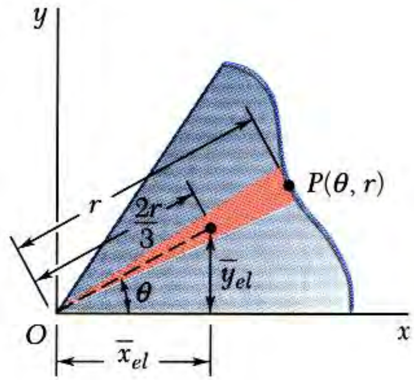
- Double integration to find the first moment may be avoided by defining *dA* as a thin rectangle or strip.



$$\bar{x}A = \int \bar{x}_{el}\, dA = \int x\left(y dx\right)$$
$$\bar{y}A = \int \bar{y}_{el}\, dA = \int \frac{y}{2}\left(y dx\right)$$

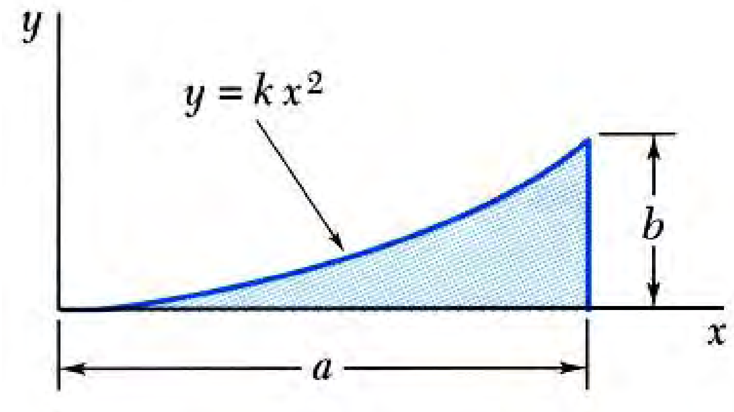


$$\bar{x}A = \int \bar{x}_{el}\, dA = \int \frac{a+x}{2}\left[(a-x)dx\right]$$
$$\bar{y}A = \int \bar{y}_{el}\, dA = \int y\left[(a-x)dx\right]$$



$$\bar{x}A = \int \bar{x}_{el}\, dA = \int \frac{2r}{3}\cos\theta\left(\frac{1}{2}r^2 d\theta\right)$$
$$\bar{y}A = \int \bar{y}_{el}\, dA = \int \frac{2r}{3}\sin\theta\left(\frac{1}{2}r^2 d\theta\right)$$

Sample Problem 5.4

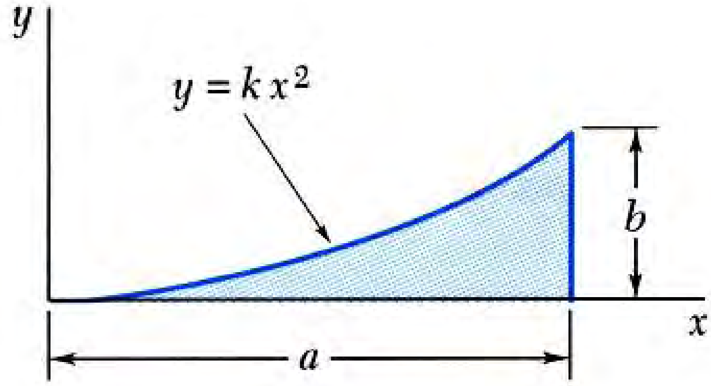


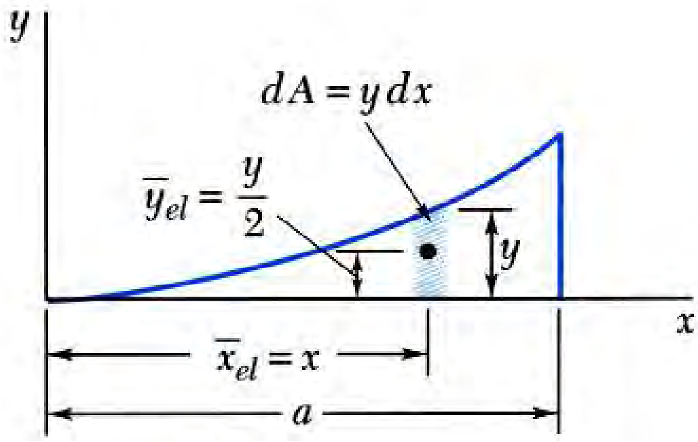
Determine by direct integration the location of the centroid of a parabolic spandrel.

<u>SOLUTION</u>:

- Determine the constant k.
- Evaluate the total area.
- Using either vertical or horizontal strips, perform a single integration to find the first moments.
- Evaluate the centroid coordinates.

Sample Problem 5.4





<u>SOLUTION</u>:

- Determine the constant k.

$$y = k\,x^2$$

$$b = k\,a^2 \Rightarrow k = \frac{b}{a^2}$$

$$y = \frac{b}{a^2}x^2 \quad or \quad x = \frac{a}{b^{1/2}}y^{1/2}$$

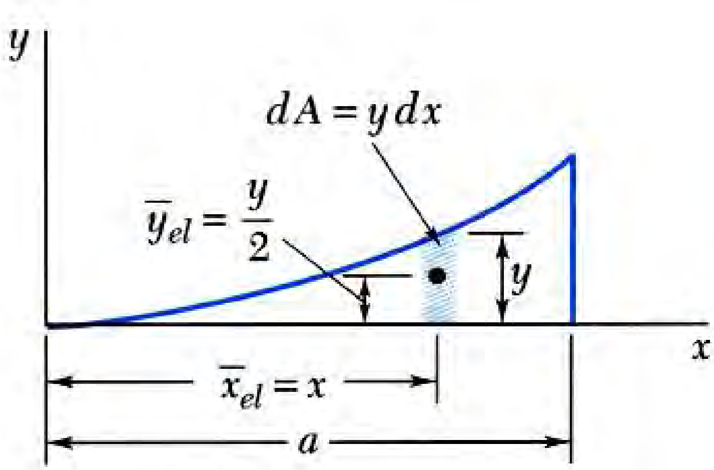
- Evaluate the total area.

$$A = \int dA$$

$$= \int y\,dx = \int_0^a \frac{b}{a^2}x^2 dx = \left[\frac{b}{a^2}\frac{x^3}{3}\right]_0^a$$

$$= \frac{ab}{3}$$

Sample Problem 5.4



- Using vertical strips, perform a single integration to find the first moments.

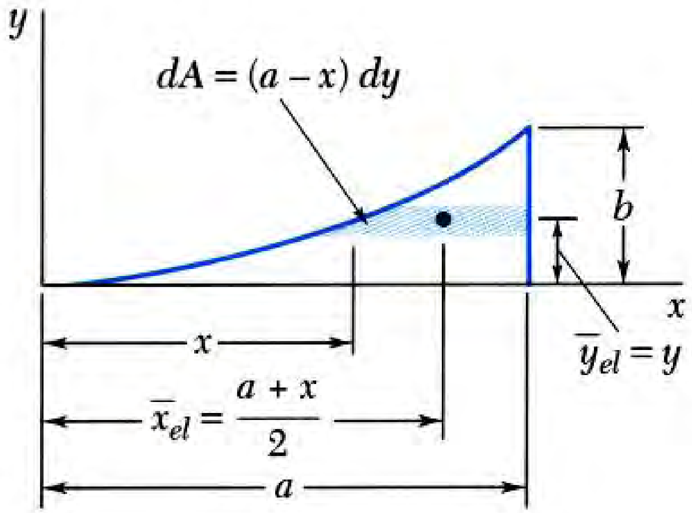
$$Q_y = \int \bar{x}_{el} dA = \int xy dx = \int_0^a x \left(\frac{b}{a^2} x^2 \right) dx$$

$$= \left[\frac{b}{a^2} \frac{x^4}{4} \right]_0^a = \frac{a^2 b}{4}$$

$$Q_x = \int \bar{y}_{el} dA = \int \frac{y}{2} y dx = \int_0^a \frac{1}{2} \left(\frac{b}{a^2} x^2 \right)^2 dx$$

$$= \left[\frac{b^2}{2a^4} \frac{x^5}{5} \right]_0^a = \frac{ab^2}{10}$$

Sample Problem 5.4



- Or, using horizontal strips, perform a single integration to find the first moments.

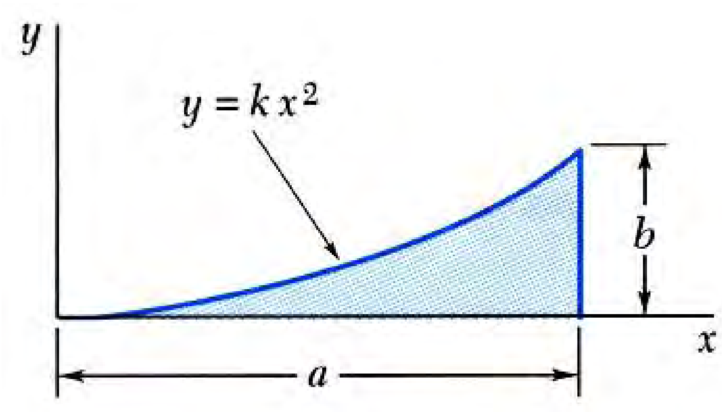
$$Q_y = \int \bar{x}_{el} dA = \int \frac{a+x}{2}(a-x)dy = \int_0^b \frac{a^2 - x^2}{2} dy$$

$$= \frac{1}{2}\int_0^b \left(a^2 - \frac{a^2}{b} y \right) dy = \frac{a^2 b}{4}$$

$$Q_x = \int \bar{y}_{el} dA = \int y(a-x)dy = \int y\left(a - \frac{a}{b^{1/2}} y^{1/2} \right) dy$$

$$= \int_0^b \left(ay - \frac{a}{b^{1/2}} y^{3/2} \right) dy = \frac{ab^2}{10}$$

Sample Problem 5.4



- Evaluate the centroid coordinates.

$$\bar{x}A = Q_y$$

$$\bar{x}\frac{ab}{3} = \frac{a^2 b}{4}$$

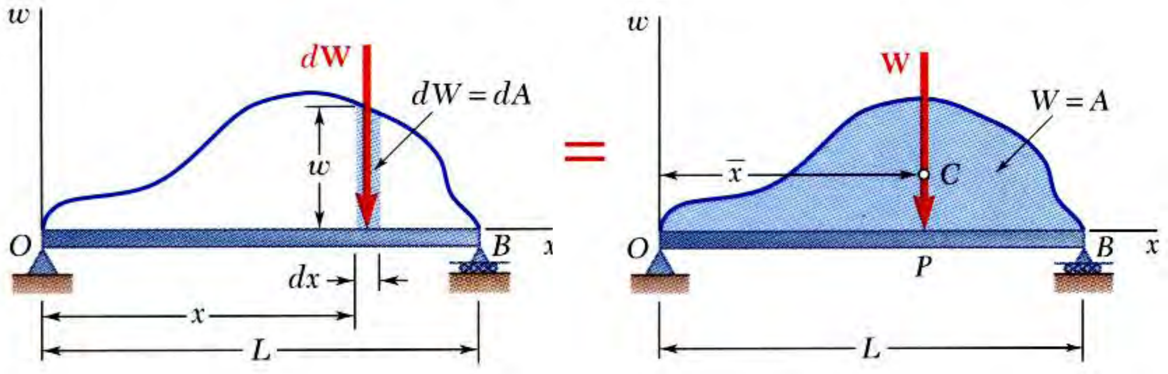
$$\boxed{\bar{x} = \frac{3}{4}a}$$

$$\bar{y}A = Q_x$$

$$\bar{y}\frac{ab}{3} = \frac{ab^2}{10}$$

$$\boxed{\bar{y} = \frac{3}{10}b}$$

Distributed Loads on Beams



$$W = \int_0^L w\,dx = \int dA = A$$

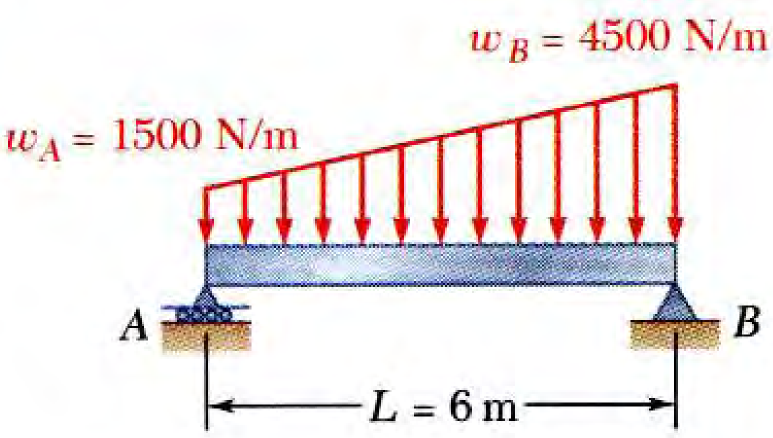
- A distributed load is represented by plotting the load per unit length, *w* (N/m) . The total load is equal to the area under the load curve.

$$(OP)W = \int x\,dW$$

$$(OP)A = \int_0^L x\,dA = \bar{x}A$$

- A distributed load can be replace by a concentrated load with a magnitude equal to the area under the load curve and a line of action passing through the area centroid.

Sample Problem 5.9

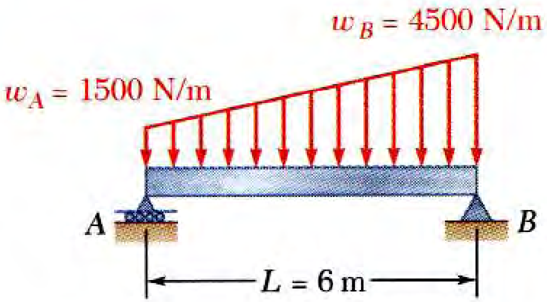


A beam supports a distributed load as shown. Determine the equivalent concentrated load and the reactions at the supports.

SOLUTION:

- The magnitude of the concentrated load is equal to the total load or the area under the curve.
- The line of action of the concentrated load passes through the centroid of the area under the curve.
- Determine the support reactions by summing moments about the beam ends.

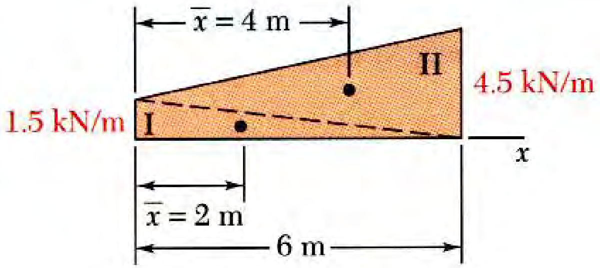
Sample Problem 5.9



SOLUTION:

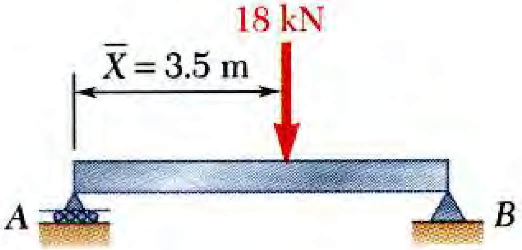
- The magnitude of the concentrated load is equal to the total load or the area under the curve.

$$F = 18.0\,\text{kN}$$



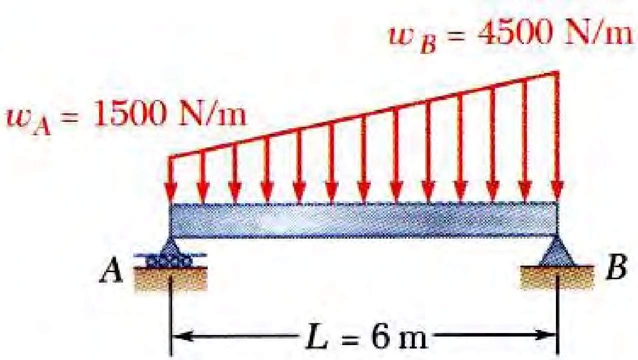
- The line of action of the concentrated load passes through the centroid of the area under the curve.

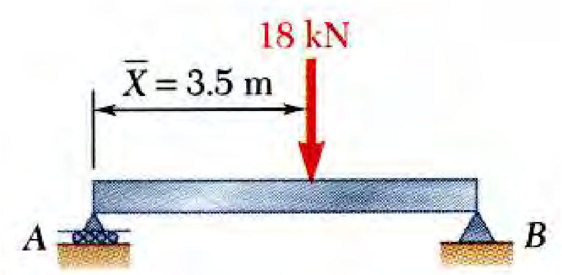
$$\bar{X} = \frac{63\,\text{kN}\cdot\text{m}}{18\,\text{kN}} \qquad \bar{X} = 3.5\,\text{m}$$



Component	A, kN	$\bar{x}$, m	$\bar{x}A$, kN · m
Triangle I	4.5	2	9
Triangle II	13.5	4	54
	$\Sigma A = 18.0$		$\Sigma \bar{x}A = 63$

Sample Problem 5.9





- Determine the support reactions by summing moments about the beam ends.

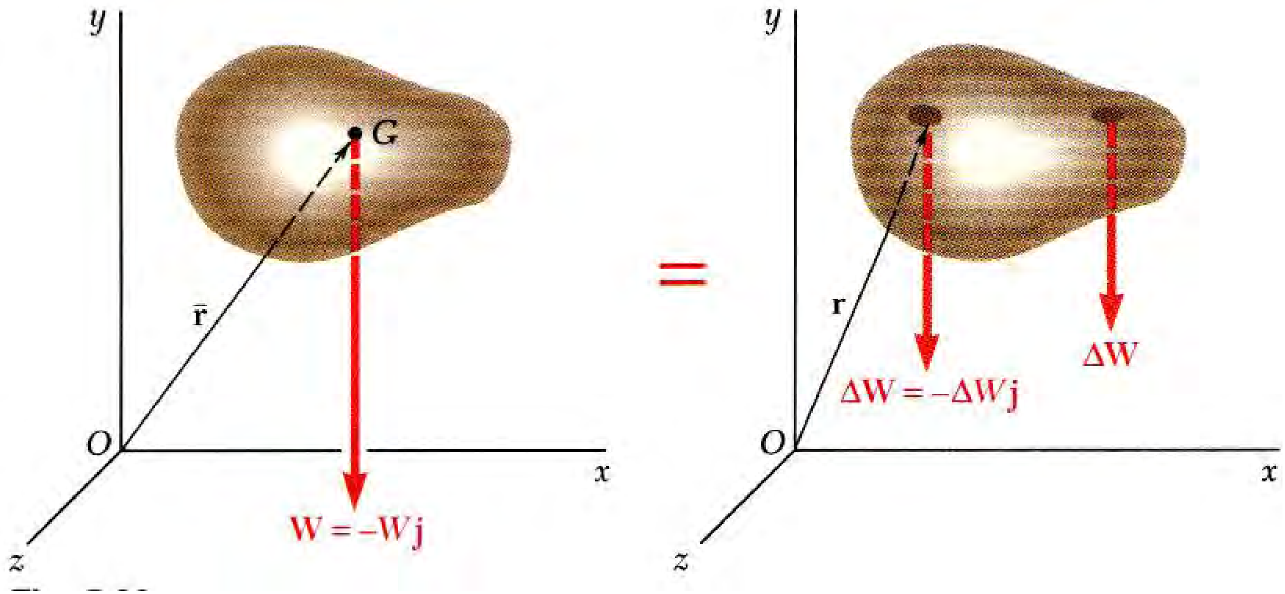
$$\sum M_A = 0: \quad B_y(6\,\text{m}) - (18\,\text{kN})(3.5\,\text{m}) = 0$$

$$\boxed{B_y = 10.5\,\text{kN}}$$

$$\sum M_B = 0: \quad -A_y(6\,\text{m}) + (18\,\text{kN})(6\,\text{m} - 3.5\,\text{m}) = 0$$

$$\boxed{A_y = 7.5\,\text{kN}}$$

Center of Gravity of a 3D Body: Centroid of a Volume



- Center of gravity *G*

$$-W\vec{j} = \sum\left(-\Delta W\vec{j}\right)$$

$$\vec{r}_G \times \left(-W\vec{j}\right) = \sum\left[\vec{r} \times \left(-\Delta W\vec{j}\right)\right]$$

$$\vec{r}_G W \times \left(-\vec{j}\right) = \left(\sum \vec{r}\Delta W\right) \times \left(-\vec{j}\right)$$

$$W = \int dW \qquad \vec{r}_G W = \int \vec{r}\, dW$$

- Results are independent of body orientation,

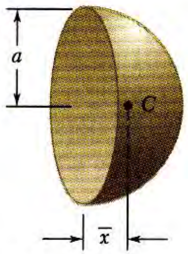
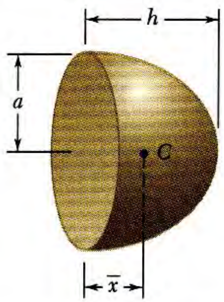
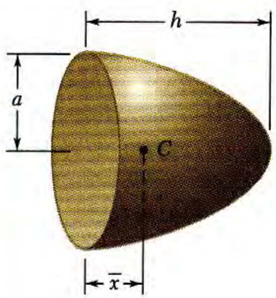
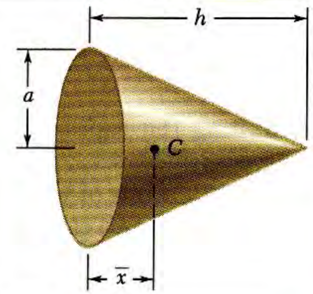
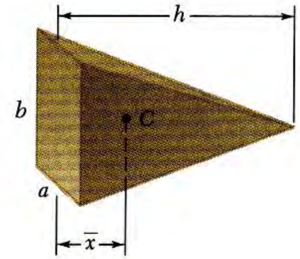
$$\bar{x}W = \int x\,dW \quad \bar{y}W = \int y\,dW \quad \bar{z}W = \int z\,dW$$

- For homogeneous bodies,

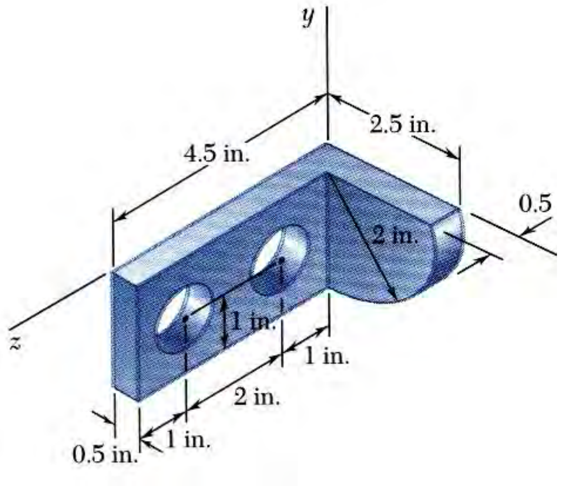
$$W = \gamma V \text{ and } dW = \gamma\, dV$$

$$\bar{x}V = \int x\,dV \quad \bar{y}V = \int y\,dV \quad \bar{z}V = \int z\,dV$$

Centroids of Common 3D Shapes

Shape		$\bar{x}$	Volume
Hemisphere		$\frac{3a}{8}$	$\frac{2}{3}\pi a^3$
Semiellipsoid of revolution		$\frac{3h}{8}$	$\frac{2}{3}\pi a^2 h$
Paraboloid of revolution		$\frac{h}{3}$	$\frac{1}{2}\pi a^2 h$
Cone		$\frac{h}{4}$	$\frac{1}{3}\pi a^2 h$
Pyramid		$\frac{h}{4}$	$\frac{1}{3}abh$

Composite 3D Bodies

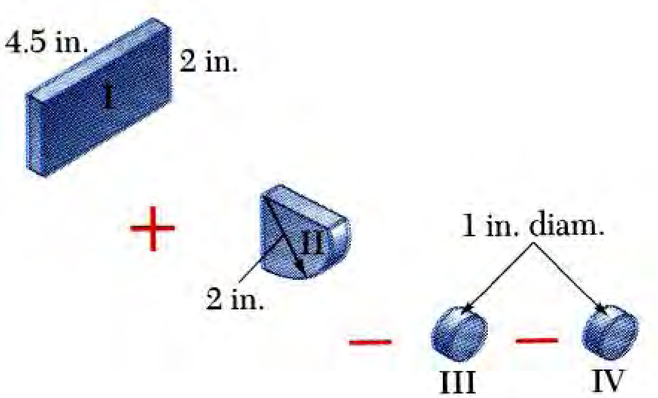


- Moment of the total weight concentrated at the center of gravity G is equal to the sum of the moments of the weights of the component parts.

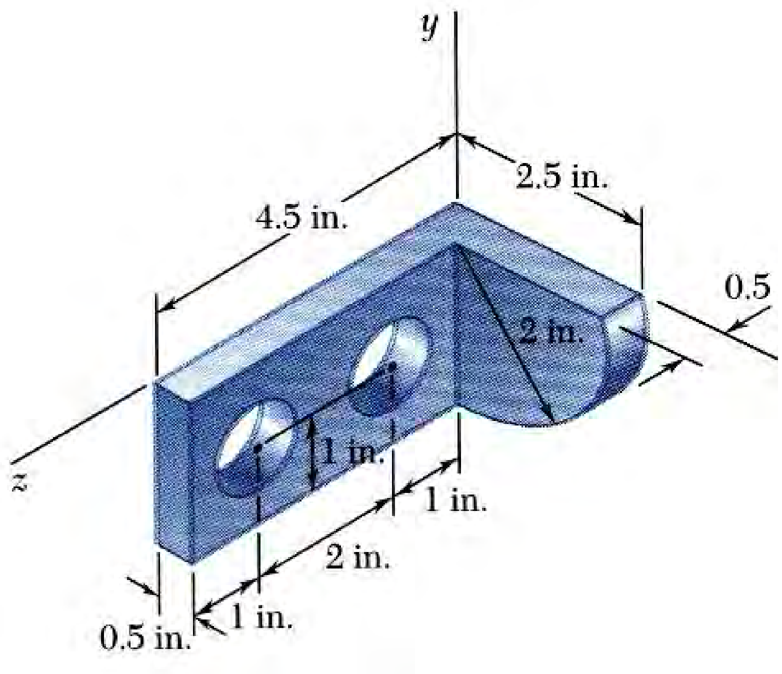
$$\bar{X}\sum W = \sum \bar{x}W \quad \bar{Y}\sum W = \sum \bar{y}W \quad \bar{Z}\sum W = \sum \bar{z}W$$

- For homogeneous bodies,

$$\bar{X}\sum V = \sum \bar{x}V \quad \bar{Y}\sum V = \sum \bar{y}V \quad \bar{Z}\sum V = \sum \bar{z}V$$



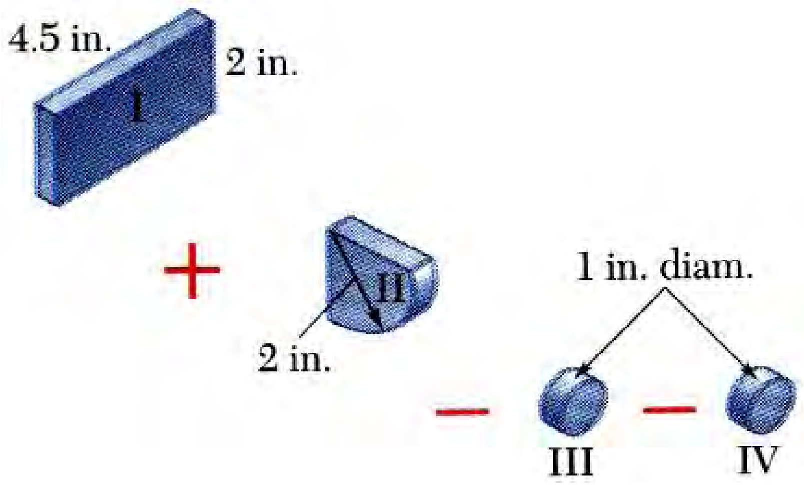
Sample Problem 5.12



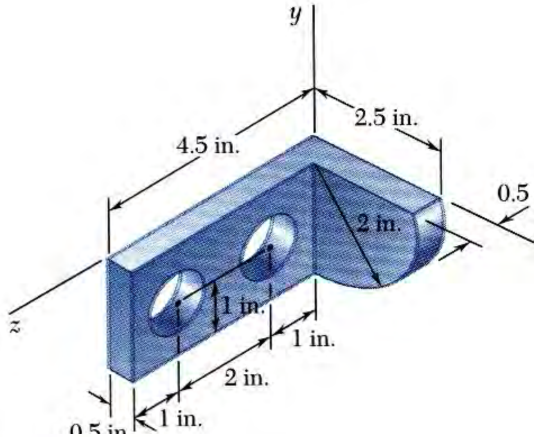
Locate the center of gravity of the steel machine element. The diameter of each hole is 1 in.

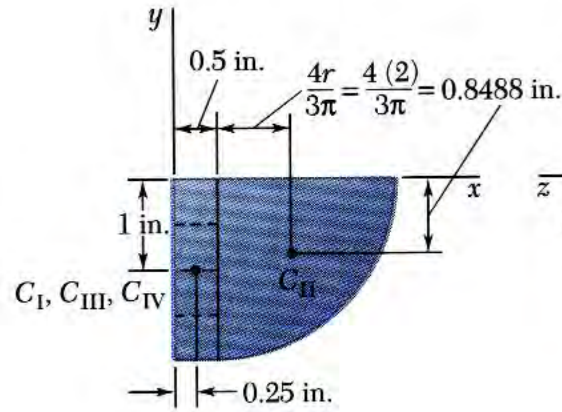
SOLUTION:

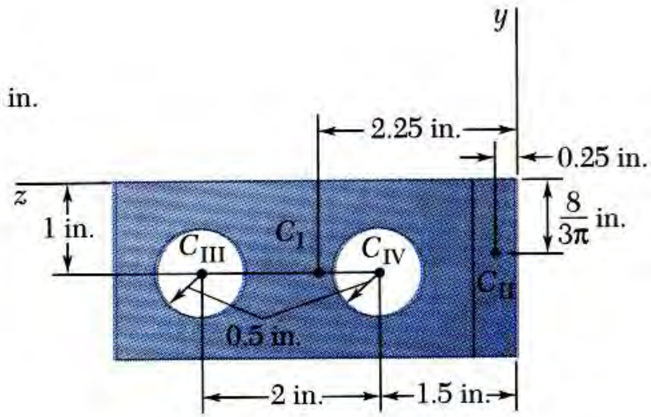
- Form the machine element from a rectangular parallelepiped and a quarter cylinder and then subtracting two 1-in. diameter cylinders.

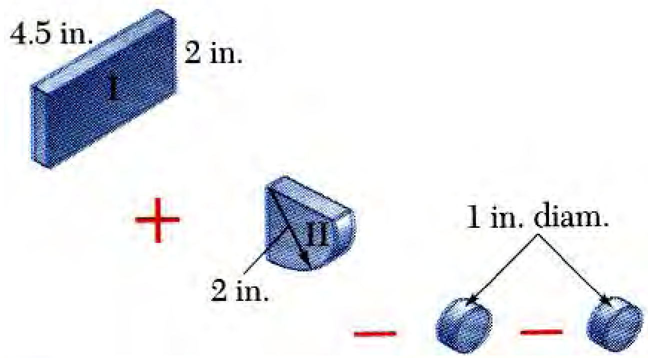


Sample Problem 5.12









	V, in³	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{z}$, in.	$\bar{x}V$, in⁴	$\bar{y}V$, in⁴	$\bar{z}V$, in⁴
I	$(4.5)(2)(0.5) = 4.5$	0.25	−1	2.25	1.125	−4.5	10.125
II	$\frac{1}{4}\pi(2)^2(0.5) = 1.571$	1.3488	−0.8488	0.25	2.119	−1.333	0.393
III	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	−1	3.5	−0.098	0.393	−1.374
IV	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	−1	1.5	−0.098	0.393	−0.589
	$\Sigma V = 5.286$				$\Sigma\bar{x}V = 3.048$	$\Sigma\bar{y}V = -5.047$	$\Sigma\bar{z}V = 8.555$

Sample Problem 5.12

	V, in³	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{z}$, in.	$\bar{x}V$, in⁴	$\bar{y}V$, in⁴	$\bar{z}V$, in⁴
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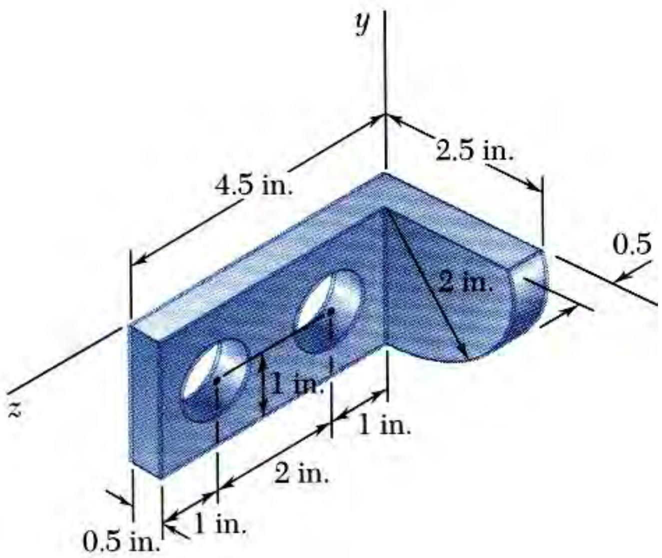

$$\bar{X} = \sum \bar{x}V / \sum V = \left(3.08\,\text{in}^4\right) / \left(5.286\,\text{in}^3\right)$$

$$\boxed{\bar{X} = 0.577\ \text{in.}}$$

$$\bar{Y} = \sum \bar{y}V / \sum V = \left(-5.047\,\text{in}^4\right) / \left(5.286\,\text{in}^3\right)$$

$$\boxed{\bar{Y} = 0.577\ \text{in.}}$$

$$\bar{Z} = \sum \bar{z}V / \sum V = \left(1.618\,\text{in}^4\right) / \left(5.286\,\text{in}^3\right)$$

$$\boxed{\bar{Z} = 0.577\ \text{in.}}$$