دانشگاه صنعتی خواجه نصیرالدین طوسی

دانشکده مهندسی هوافضا

کنترل اتوماتیک (مروری بر تبدیل لاپلاس و جبر ماتریس ها)

استاد درس: مرتضی طایفی

$$s = \sigma + j\omega$$ Complex variable متغیر مختلط

Real part مقدار حقیقی　　Imaginary part مقدار موهومی

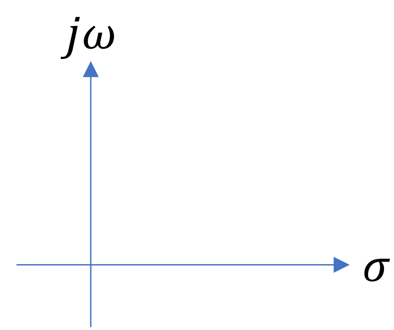
$$G(s) = G_x + jG_y$$ Complex function تابع مختلط

G_x and G_y are real quantities

magnitude of $G(s)$ is $\sqrt{G_x^2 + G_y^2}$

angle θ of $G(s)$ is $\tan^{-1}(G_y/G_x)$

complex conjugate of $G(s)$ is $G(s) = G_x - jG_y$



S plane

صفر و قطب یک تابع مختلط

- Ordinary point

  Points in the S plane at which the function G(s) is analytic

- Singular point نقاط منفرد

  Points in the S plane at which the function G(s) is not analytic

- **Pole قطب**

  Singular points at which the function G(s) or its derivatives approach infinity ریشه های چند جمله ای مخرج

- **Zero صفر**

  Singular points at which the function G(s) equals zero

  ریشه های چند جمله ای صورت

مثال

$$G(s) = \frac{K(s+2)(s+10)}{s(s+1)(s+5)(s+15)^2}$$

$G(s)$ has zeros at $s = -2$, $s = -10$,

simple poles at $s = 0$, $s = -1$, $s = -5$,

a double pole (multiple pole of order 2) at $s = -15$.

for large values of s $\quad G(s) \doteqdot \frac{K}{s^3}$

$G(s)$ possesses a triple zero (multiple zero of order 3) at $s = \infty$.

تابع مختلط فوق دارای ۵ صفر و ۵ قطب می باشد.

تبدیل لاپلاس Laplace Transform

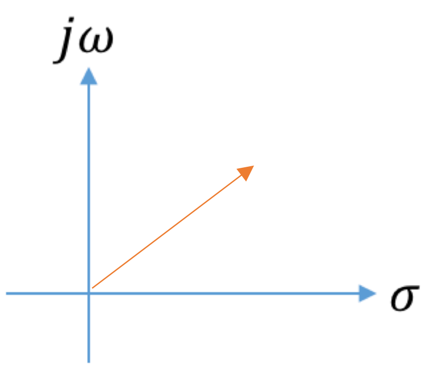
$f(t)$ = a function of time t such that $f(t) = 0$ for $t < 0$

s = a complex variable

$F(s)$ = Laplace transform of $f(t)$

Laplace transform of $f(t)$ is given by

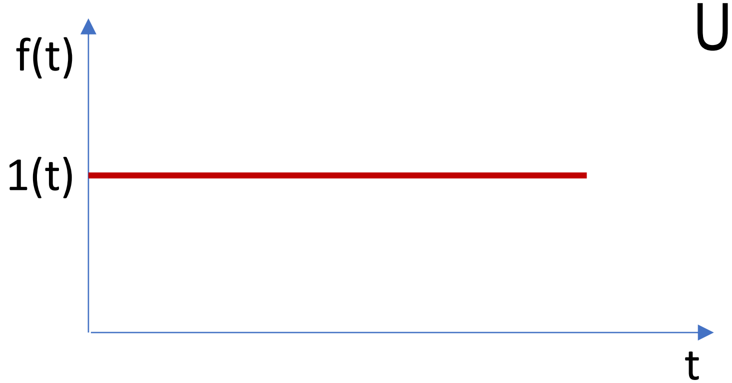
$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st}\,dt[f(t)] = \int_0^\infty f(t)e^{-st}\,dt$$



رابطه اویلر:

$e^{j\theta} = \cos\theta + j\sin\theta$ تابع نمایی مختلط

Unit Step function **تابع پله واحد**

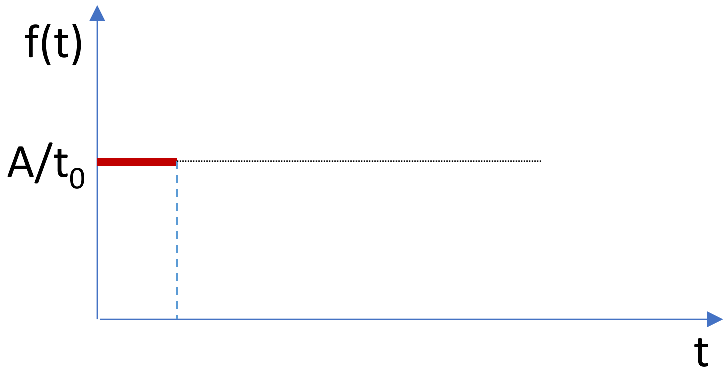


$$\mathscr{L}[f(t)] = \int_0^{+\infty} 1 \times e^{-st} dt = \frac{-1}{s} e^{-st} \Big|_0^{-\infty} = \frac{-1}{s}\left(e^{-\infty} - e^{0}\right) = \frac{-1}{s}(0-1) = \frac{1}{s}$$

عملکرد پله واحد در ایجاد تاخیر در یک تابع

$$\mathscr{L}[f(t-\alpha)1(t-\alpha)] = e^{-\alpha s}F(s) \qquad \alpha \geq 0$$

Pulse function تابع پالس



$$f(t) = \frac{A}{t_0}1(t) - \frac{A}{t_0}1(t - t_0)$$

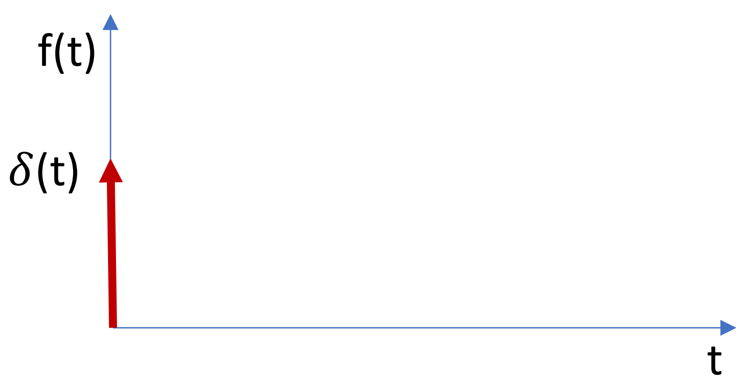
$$\mathscr{L}[f(t)] = \frac{A}{t_0 s} - \frac{A}{t_0 s}e^{-st_0}$$

ثابت A

$t_0 \rightarrow 0$

$$\begin{aligned} g(t) &= \lim_{t_0 \to 0} \frac{A}{t_0}, && \text{for } 0 < t < t_0 \\ &= 0, && \text{for } t < 0, t_0 < t \end{aligned}$$

Impulse function تابع ضربه



$$\begin{aligned} \mathscr{L}[g(t)] &= \lim_{t_0 \to 0}\left[\frac{A}{t_0 s}(1 - e^{-st_0})\right] \\ &= \lim_{t_0 \to 0} \frac{\frac{d}{dt_0}[A(1 - e^{-st_0})]}{\frac{d}{dt_0}(t_0 s)} \\ &= \frac{As}{s} = A \end{aligned}$$

تبدیل لاپلاس توابع معروف (جدول کامل در کتاب)

$f(t)$	$F(s)$
ضربه واحد Unit impulse $\delta(t)$	1
پله واحد Unit step $1(t)$	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$t^n \qquad (n = 1, 2, 3, \ldots)$	$\frac{n!}{s^{n+1}}$

e^{-at}	$\frac{1}{s+a}$
te^{-at}	$\frac{1}{(s+a)^2}$
$t^n e^{-at} \qquad (n = 1, 2, 3, \ldots)$	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

برخی روابط تبدیل لاپلاس

(لیست کامل روابط در کتاب)

$$\mathcal{L}\left[f\left(\frac{1}{a}\right)\right] = aF(as)$$

تابع نمایی کاهشی

$$\mathcal{L}[e^{-\alpha t}f(t)] = F(s+a)$$

$$\mathcal{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$$

ناشی از خطی بودن تبدیل لاپلاس

$$\mathcal{L}[Af(t)] = AF(s)$$

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$$

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s) \qquad (n = 1, 2, 3, \dots)$$

مشتق گیری از تبدیل لاپلاس

$$\mathcal{L}_{\pm}\left[\frac{d^n}{dt^n}f(t)\right] = s^n F(s) - \sum_{k=1}^{n} s^{n-k} \overset{(k-1)}{f}(0\pm)$$

تبدیل لاپلاس مشتق تابع در حوزه زمان

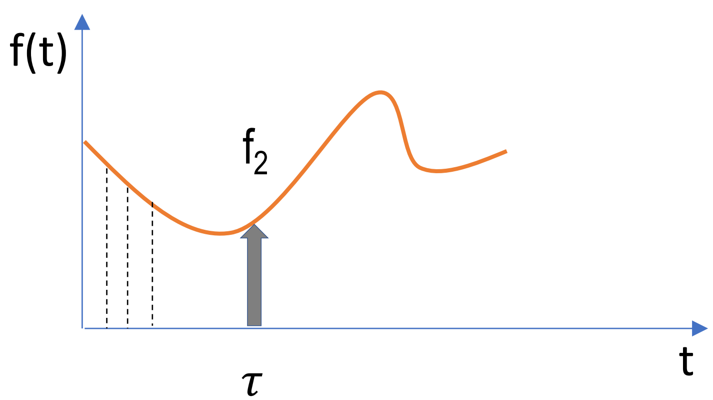
شرایط اولیه

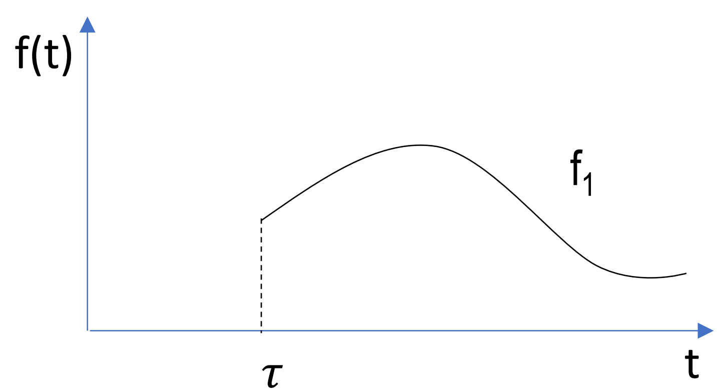
انتگرال کانولوشن Convolution Integral

حاصلضرب در حوزه مختلط معادل کانولوشن در حوزه زمان است.

$$\mathcal{L}\left[\int_0^t f_1(t-\tau)f_2(\tau)\,d\tau\right] = F_1(s)F_2(s)$$

در واقع این انتگرال بیان کننده پاسخ یک سیستم **LTI** به یک ورودی عمومی f_2 می باشد. f_1 پاسخ سیستم به ضربه واحد می باشد.





قضیه مقدار نهایی Final Value Theorem

در صورتی که تمام قطب های sF(s) در سمت چپ s plane قرار گیرند، داریم:

$$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

برای محاسبه پاسخ حالت ماندگار سیستم می تواند استفاده بشود.

قضیه مقدار اولیه Initial Value Theorem

$$f(0+) = \lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s)$$

برای پیدا کردن مقدار اولیه می تواند استفاده بشود.

بسط کسرهای جزیی Partial-Fraction Expansion

تابع مختلط F(s):

$$F(s) = \frac{B(s)}{A(s)} = \frac{K(s + z_1)(s + z_2)\cdots(s + z_m)}{(s + p_1)(s + p_2)\cdots(s + p_n)}, \qquad \text{for } m < n$$

where $p_1, p_2, \ldots, p_n$ and $z_1, z_2, \ldots, z_m$ are either real or complex quantities,

در صورتی که فقط قطب های ساده (مرتبه یک) مجزا داشته باشیم:

$$F(s) = \frac{B(s)}{A(s)} = \frac{a_1}{s + p_1} + \frac{a_2}{s + p_2} + \cdots + \frac{a_n}{s + p_n}$$

محاسبه ضریب پسماند یا Residue برای قطب k ام:

$$a_k = \left[(s + p_k) \frac{B(s)}{A(s)} \right]_{s=-p_k}$$

با بسط کسرهای جزیی، محاسبه f(t) راحت می شود:

$$\mathcal{L}^{-1}\left[\frac{a_k}{s + p_k} \right] = a_k e^{-p_k t}$$

$$f(t) = \mathcal{L}^{-1}[F(s)] = a_1 e^{-p_1 t} + a_2 e^{-p_2 t} + \cdots + a_n e^{-p_n t}, \qquad \text{for } t \geq 0$$

مثال:

$$F(s) = \frac{s + 3}{(s + 1)(s + 2)} = \frac{a_1}{s + 1} + \frac{a_2}{s + 2} \longrightarrow f(t) = 2e^{-t} - e^{-2t}$$

در صورت وجود **قطب های مزدوج مختلط**، راحت تر است که از روش معمول شرح داده شده استفاده نشود.

مثال:

$$F(s) = \frac{2s + 12}{s^2 + 2s + 5}$$

به جای بسط روبرو

$$s^2 + 2s + 5 = (s + 1 + j2)(s + 1 - j2)$$

از طریق زیر عمل می کنیم. ابتدا سعی میکنیم تا ترم های مشابه زیر را ایجاد کنیم

$$\mathcal{L}[e^{-\alpha t} \sin \omega t] = \frac{\omega}{(s + \alpha)^2 + \omega^2}$$

$$\mathcal{L}[e^{-\alpha t} \cos \omega t] = \frac{s + \alpha}{(s + \alpha)^2 + \omega^2}$$

برای مثال مورد نظر داریم:

$$F(s) = \frac{2s + 12}{s^2 + 2s + 5} = \frac{10 + 2(s + 1)}{(s + 1)^2 + 2^2}$$

$$= 5\frac{2}{(s + 1)^2 + 2^2} + 2\frac{s + 1}{(s + 1)^2 + 2^2}$$

$$f(t) = \mathscr{L}^{-1}[F(s)]$$

$$= 5\mathscr{L}^{-1}\left[\frac{2}{(s + 1)^2 + 2^2}\right] + 2\mathscr{L}^{-1}\left[\frac{s + 1}{(s + 1)^2 + 2^2}\right]$$

$$= 5e^{-t}\sin 2t + 2e^{-t}\cos 2t, \qquad \text{for } t \geq 0$$

بسط کسرهای جزیی برای **تابع با قطب تکراری** (مرتبه بالاتر از یک)

مثال:

$$F(s) = \frac{s^2 + 2s + 3}{(s+1)^3}$$

$$F(s) = \frac{B(s)}{A(s)} = \frac{b_1}{s+1} + \frac{b_2}{(s+1)^2} + \frac{b_3}{(s+1)^3}$$

$$\begin{aligned} b_3 &= \left[(s+1)^3 \frac{B(s)}{A(s)} \right]_{s=-1} \\ &= (s^2 + 2s + 3)_{s=-1} \\ &= 2 \end{aligned}$$

$$\begin{aligned} b_2 &= \left\{ \frac{d}{ds} \left[(s+1)^3 \frac{B(s)}{A(s)} \right] \right\}_{s=-1} \\ &= \left[\frac{d}{ds} (s^2 + 2s + 3) \right]_{s=-1} \\ &= (2s + 2)_{s=-1} \\ &= 0 \end{aligned}$$

$$b_1 = \frac{1}{2!}\left\{\frac{d^2}{ds^2}\left[(s+1)^3\frac{B(s)}{A(s)}\right]\right\}_{s=-1}$$

$$\frac{1}{(k-j)!}\lim_{s\to -p_i}\frac{d^{k-j}}{ds^{k-j}}\left(\left(s+p_i\right)^k F(s)\right)$$

K=3, j=1

$$= \frac{1}{2!}\left[\frac{d^2}{ds^2}(s^2+2s+3)\right]_{s=-1}$$

$$= \frac{1}{2}(2) = 1$$

$$f(t) = \mathcal{L}^{-1}[F(s)]$$

$$= \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \mathcal{L}^{-1}\left[\frac{0}{(s+1)^2}\right] + \mathcal{L}^{-1}\left[\frac{2}{(s+1)^3}\right]$$

$$= e^{-t} + 0 + t^2e^{-t}$$

$$= (1+t^2)e^{-t}, \qquad \text{for } t \geq 0$$

توجه شود که هر تابع تبدیل سیستم های دینامیکی قابل بسط به سه نوع توابع ذکر شده میباشد.

حالتی که درجه صورت بیشتر از مخرج می باشد، ابتدا صورت به مخرج تقسیم می شود.

مثال:

$$G(s) = \frac{s^3 + 5s^2 + 9s + 7}{(s+1)(s+2)}$$

$$G(s) = s + 2 + \frac{s+3}{(s+1)(s+2)}$$

مثال قبل

$$g(t) = \frac{d}{dt}\delta(t) + 2\delta(t) + 2e^{-t} - e^{-2t}, \qquad \text{for } t \geq 0-$$

Partial-Fraction Expansion with MATLAB.

$$\frac{B(s)}{A(s)} = \frac{\text{num}}{\text{den}} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

$$\text{num} = [b_0 \ \ b_1 \ \ \ldots \ \ b_n]$$
$$\text{den} = [1 \ \ a_1 \ \ \ldots \ \ a_n]$$

The command

```
[r,p,k] = residue(num,den)
```

$$\frac{B(s)}{A(s)} = \frac{r(1)}{s - p(1)} + \frac{r(2)}{s - p(2)} + \cdots + \frac{r(n)}{s - p(n)} + k(s)$$

$$\frac{B(s)}{A(s)} = \frac{2s^3 + 5s^2 + 3s + 6}{s^3 + 6s^2 + 11s + 6}$$

```
[r,p,k] = residue(num,den)

r =

   -6.0000
   -4.0000
    3.0000

p =

   -3.0000
   -2.0000
   -1.0000

k =

     2
```

$$\frac{B(s)}{A(s)} = \frac{s^2 + 2s + 3}{(s+1)^3} = \frac{s^2 + 2s + 3}{s^3 + 3s^2 + 3s + 1}$$

```
num = [1  2  3];
den = [1  3  3  1];
[r,p,k] = residue(num,den)

r =

    1.0000
    0.0000
    2.0000

p =

   -1.0000
   -1.0000
   -1.0000

k =

    []
```

Matrices and Vectors

$$\mathbf{A} = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

$m \times n$ is called the **size** of the matrix.

If $m = n$, we call $\mathbf{A}$ an $n \times n$ **square matrix**.

$a_{11}, a_{22}, \cdots, a_{nn}$ is called the **main diagonal** of $\mathbf{A}$.

A **vector** is a matrix with only one row or column.

Addition of Matrices

The **sum** of two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ ***of the same size*** is written $\mathbf{A} + \mathbf{B}$ and has the entries $a_{jk} + b_{jk}$ obtained by adding the corresponding entries of $\mathbf{A}$ and $\mathbf{B}$. Matrices of different sizes cannot be added.

Scalar Multiplication (Multiplication by a Number)

The **product** of any $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ and any **scalar** c (number c) is written $c\mathbf{A}$ and is the $m \times n$ matrix $c\mathbf{A} = [ca_{jk}]$ obtained by multiplying each entry of $\mathbf{A}$ by c.

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$$

$$\mathbf{A} + \mathbf{0} = \mathbf{A}$$

$$\mathbf{A} + (-\mathbf{A}) = \mathbf{0}.$$

$$c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$$

$$(c + k)\mathbf{A} = c\mathbf{A} + k\mathbf{A}$$

$$c(k\mathbf{A}) = (ck)\mathbf{A}$$

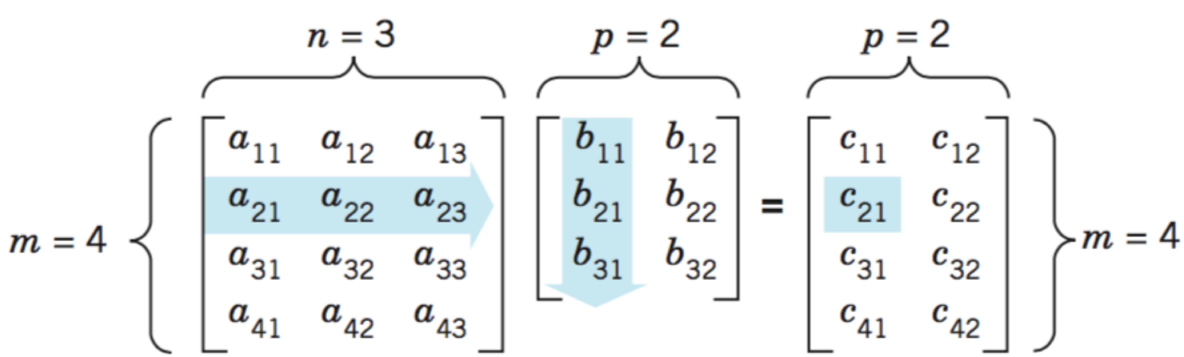
$$1\mathbf{A} = \mathbf{A}.$$

Multiplication of a Matrix by a Matrix

The **product** $\mathbf{C} = \mathbf{AB}$ (in this order) of an $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ times an $r \times p$ matrix $\mathbf{B} = [b_{jk}]$ is defined if and only if $r = n$ and is then the $m \times p$ matrix $\mathbf{C} = [c_{jk}]$ with entries

$$\textbf{(1)} \qquad c_{jk} = \sum_{l=1}^{n} a_{jl}b_{lk} = a_{j1}b_{1k} + a_{j2}b_{2k} + \cdots + a_{jn}b_{nk} \qquad \begin{array}{l} j = 1, \cdots, m \\ k = 1, \cdots, p. \end{array}$$

$$\begin{array}{cccc} \mathbf{A} & \mathbf{B} & = & \mathbf{C} \\ [m \times n] & [n \times p] & = & [m \times p]. \end{array}$$



Notations in a product **AB** = **C**

Transposition of Matrices and Vectors

The transpose of an $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ is the $n \times m$ matrix $\mathbf{A}^\mathsf{T}$ (read *A transpose*) that has the first *row* of $\mathbf{A}$ as its first *column*, the second *row* of $\mathbf{A}$ as its second *column*, and so on. Thus the transpose of $\mathbf{A}$ in (2) is $\mathbf{A}^\mathsf{T} = [a_{kj}]$, written out

$$\mathbf{A}^\mathsf{T} = [a_{kj}] = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \cdot & \cdot & \cdots & \cdot \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}. \tag{9}$$

As a special case, transposition converts row vectors to column vectors and conversely.

$$(\mathbf{A}^\mathsf{T})^\mathsf{T} = \mathbf{A}$$

$$(\mathbf{A} + \mathbf{B})^\mathsf{T} = \mathbf{A}^\mathsf{T} + \mathbf{B}^\mathsf{T}$$

$$(c\mathbf{A})^\mathsf{T} = c\mathbf{A}^\mathsf{T}$$

$$(\mathbf{AB})^\mathsf{T} = \mathbf{B}^\mathsf{T}\mathbf{A}^\mathsf{T}.$$

$$\mathbf{A}^\mathsf{T} = \mathbf{A} \qquad (\text{thus } a_{kj} = a_{jk}),$$

Symmetric Matrix

$$\mathbf{A}^\mathsf{T} = -\mathbf{A} \qquad (\text{thus } a_{kj} = -a_{jk}, \text{ hence } a_{jj} = 0).$$

Skew-Symmetric Matrix

Linear Independence and Dependence of Vectors

Given any set of m vectors $\mathbf{a}_{(1)}, \cdots, \mathbf{a}_{(m)}$

$$c_1\mathbf{a}_{(1)} + c_2\mathbf{a}_{(2)} + \cdots + c_m\mathbf{a}_{(m)} = \mathbf{0}.$$

all c_j's zero → If this is the only m-tuple of scalars

$\mathbf{a}_{(1)}, \cdots, \mathbf{a}_{(m)}$ are said to form a *linearly independent set*

Linear Dependence of Vectors

Consider p vectors each having n components. If $n < p$, then these vectors *are linearly dependent.*

Rank of a Matrix

The **rank** of a matrix $\mathbf{A}$ is the maximum number of linearly independent row vectors of $\mathbf{A}$. It is denoted by rank $\mathbf{A}$.

Rank in Terms of Column Vectors

The rank r of a matrix $\mathbf{A}$ *equals the maximum number of linearly independent* ***column*** *vectors of* $\mathbf{A}$.

Hence $\mathbf{A}$ *and its transpose* $\mathbf{A}^{\mathsf{T}}$ *have the same rank.*

A **determinant of second order** is denoted and defined by

$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}. \tag{1}$$

A **determinant of third order** can be defined by

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}. \tag{4}$$

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \qquad (j = 1, 2, \cdots, \text{or } n)$$

M_{jk} is called the **minor** *of* a_{jk} *in* D, The **inverse** of an $n \times n$ matrix $\mathbf{A} = [a_{jk}]$ is denoted by $\mathbf{A}^{-1}$ and is an $n \times n$ matrix such that

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \tag{1}$$

where $\mathbf{I}$ is the $n \times n$ unit matrix (see Sec. 7.2).

If $\mathbf{A}$ has an inverse, then $\mathbf{A}$ is called a **nonsingular matrix**. If $\mathbf{A}$ has no inverse, then $\mathbf{A}$ is called a **singular matrix**.

Existence of the Inverse

The inverse $\mathbf{A}^{-1}$ *of an* $n \times n$ *matrix* $\mathbf{A}$ *exists if and only if* rank $\mathbf{A} = n$, *thus* (by Theorem 3, Sec. 7.7) *if and only if* det $\mathbf{A} \neq 0$. *Hence* $\mathbf{A}$ *is nonsingular if* rank $\mathbf{A} = n$, *and is singular if* rank $\mathbf{A} < n$.

The proof will also show that $\mathbf{Ax} = \mathbf{b}$ implies $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$

Inverse of a Matrix by Determinants

The inverse of a nonsingular $n \times n$ *matrix* $\mathbf{A} = [a_{jk}]$ *is given by*

$$\text{(4)} \qquad \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}[C_{jk}]^{\mathsf{T}} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \cdot & \cdot & \cdots & \cdot \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix},$$

$$C_{jk} = (-1)^{j+k} M_{jk}$$

In particular, the inverse of

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \textit{is} \qquad \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

MATLAB Approach to Obtain the Inverse of a Square Matrix. The inverse of a square matrix **A** can be obtained with the command

```
inv(A)
```

For example, if matrix **A** is given by

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & 0 \\ 1 & 2 & 5 \end{bmatrix}$$

then the inverse of matrix **A** is obtained as follows:

```
A = [1  1  2;3  4  0;1  2  5];
inv(A)

ans =

     2.2222    -0.1111    -0.8889
    -1.6667     0.3333     0.6667
     0.2222    -0.1111     0.1111
```

$$\mathbf{AB} \neq \mathbf{BA}.$$

$$(\mathbf{A}^{-1})^{-1} = \mathbf{A}.$$

$$(\mathbf{AC})^{-1} = \mathbf{C}^{-1}\mathbf{A}^{-1}.$$

$$\det(\mathbf{AB}) = \det(\mathbf{BA}) = \det \mathbf{A} \det \mathbf{B}.$$

If rank $\mathbf{A} = n$ *and* $\mathbf{AB} = \mathbf{AC}$, *then* $\mathbf{B} = \mathbf{C}$.

Eigenvalues and Eigenvectors

Consider the following vector equation:

$$\mathbf{Ax} = \lambda\mathbf{x}.$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}.$$

Eigenvalues

The eigenvalues of a square matrix **A** *are the roots of the characteristic equation* (4) *of* **A**.

Hence an $n \times n$ *matrix has at least one eigenvalue and at most* n *numerically different eigenvalues.*

(4) $$D(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0.$$

Eigenvalue Problem

Find the eigenvalues and eigenvectors of the matrix $\mathbf{A} = \begin{bmatrix} -4.0 & 4.0 \\ -1.6 & 1.2 \end{bmatrix}$

Solution. The characteristic equation is the quadratic equation

$$\det|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} -4-\lambda & 4 \\ -1.6 & 1.2-\lambda \end{vmatrix} = \lambda^2 + 2.8\lambda + 1.6 = 0.$$

It has the solutions $\lambda_1 = -2$ and $\lambda_2 = -0.8$. These are the eigenvalues of $\mathbf{A}$.

For $\lambda = \lambda_1 = -2$ we have

$$\begin{aligned} (-4.0 + 2.0)x_1 + \quad 4.0x_2 &= 0 \\ -1.6x_1 + (1.2 + 2.0)x_2 &= 0. \end{aligned}$$

A solution of the first equation is $x_1 = 2, x_2 = 1$.

Eigenvectors are $\mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Similarly, $\mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ 0.8 \end{bmatrix}$