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An interval solution for the n^{th} order linear ODEs with interval initial conditions

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Abstract. In this paper, a new method for interval solution of the n^{th} order linear ordinary differential equations (ODEs) with interval initial conditions is constructed. In this approach, by using the Neher's algorithm [16], first we obtain a guaranteed enclosure solution for an initial point value problem and then based on the Moore's idea [7, 13], we transform this solution to arrive at an interval solution for the main problem. For the sake of clarity, we present an algorithm in terms of the linear second order ODEs ($n = 2$). Finally, some numerical examples are presented to demonstrate the efficiency of the proposed algorithm.

AMS subject classifications: 65D10, 92C45

Key words: interval method, linear n^{th} order ordinary differential equations, interval initial value problem, verified solution, guaranteed error bound

1. Introduction

In a computer with a floating point number system, an approximate solution of the initial value problems (IVPs), suffers from discretization and round off errors. One of the ways which can use floating-point arithmetic and consider rounding errors and still be able to provide guaranteed results is verified computations. Also one of the powerful tools for verified computations is interval arithmetic. In interval arithmetic, operations between intervals are employed to calculate guaranteed bounds which contain discretization and round off errors.

Interval methods for ODEs provide a natural approach for computing the desired enclosure for cases with uncertain initial values those can be expressed as intervals. Classical approximate solution methods for ODEs are not generally useful because, in essence, an infinity of system solution would be needed to determine the enclosure.

A serious problem with interval methods is the overestimation of bounds, caused by the dependency problem of the interval arithmetic and the wrapping effect. There are several methods for reducing the overestimation of bounds in literature. For example, AWA package [11] uses a QR-factorization method [13] and VNODE package [11] uses QR-factorization with an interval Hermite-Obreschkoff method [10]. Some well known methods, e.g. Taylor based models for reducing the dependency

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problem and the wrapping effect have also been described by Berz and Makino in [17, 1, 2, 3, 5, 6].

Available interval ODE solvers concentrate on the uncertainties in the initial values. Some solvers, including AWA and VNODE, can take interval parameters as input. However, because of the wrapping effect, these solvers can be defective due to the size of the enclosure, which can grow so quickly that it causes the integration to terminate.

In this paper, we propose an interval based method for the verified integration of ODEs as:

$$\begin{cases} y^{(n)} = \sum_{i=0}^{n-1} p_i(t)y^{(i)} + p_{-1}(t), & t \in [0,r], \\ y^{(i)}(0) \in Y_{i0} = [\underline{y}_{i0}, \overline{y}_{i0}], & i = 0,1,\dots,n-1, \end{cases} \tag{1}$$

where $\underline{y}_{i0}, \overline{y}_{i0} \in \mathbb{R}$, and the functions $p_i(t)$ are assumed to be analytic with power series expansion:

$$p_i(t) = \sum_{j=0}^{\infty} b_{ij} t^j, \quad i = 0,1,\dots,n-1, \quad t \in [0,r].$$

It is well known that ODEs with real points initial conditions have a real solution $y(t)$ in terms of a power series. In [16], Neher has offered an enclosure method for the solution of (1) with point initial conditions, that computes an interval solution Y_1 such that $y(h) \in Y_1$, for $0 < h \leq r$, which guarantees all roundoff and truncation errors. However, this method only solves problems with real point initial conditions. In addition, there is only one step in the Neher's algorithm which is sometimes large and this large step size in the integration has the side-effect in the convergence of the Taylor series. Hence, any implementation of the Neher's algorithm should be based on a double precision arithmetic. In [16], Neher has mentioned that intervals for initial conditions are not allowed in his algorithm. Therefore, we cannot continue the method after the first step, because in the next steps the initial conditions are intervals. However, if we could get intervals as initial conditions in the Neher's algorithm, we would be able to take more than one step with small step sizes.

In this paper, we present an algorithm that enables us to continue the Neher's algorithm with more steps for a numerical solution to a general class of linear n^{th} order ODEs with interval initial conditions.

The paper is organized as follows. Section 2 introduces some preliminary results and notations of interval analysis. In Section 3, we introduce and analyze the proposed method. In Section 4, the special case $n = 2$ is considered to carry out a detailed description of the method and the proposed algorithm. Finally in Section 5, we present some examples and their numerical solutions to demonstrate the accuracy and efficiency of the proposed method.

2. Some preliminary notations

Throughout this paper, an interval means a bounded and closed set as

$$X = [\underline{X}, \overline{X}] = \{x : \underline{X} \leq x \leq \overline{X}\}.$$

Following [8, 9], the four elementary operations of real arithmetic can be extended to intervals. Operations over intervals $\diamond \in \{+, -, \times, \div\}$ are defined by the general rule as:

$$X \diamond Y = \{x \diamond y | x \in X, y \in Y\}.$$

Further details regarding the interval arithmetic may be found in [13, 8, 9].

The midpoint and the width of X are denoted by $m(X) := (\underline{X} + \overline{X})/2$ and $w(X) := \overline{X} - \underline{X}$, respectively. Now, let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function on $D \subset \mathbb{R}^n$, we consider functions whose representations contain only a finite number of constants, variables, arithmetic operations, and standard functions. By interval-arithmetic evaluation of f on X, which we denote by $f(X)$, we mean replacing each occurrence of a real variable with the corresponding interval, by replacing the standard functions with enclosures of their ranges, and by performing interval-arithmetic operations instead of the real operations.

Consider a grid $\{0 = t_0 < t_1 < \cdots < t_m = T\}$, which is not necessarily equally spaced, and denote the step size from t_j to t_{j+1} by $h_j = t_{j+1} - t_j$. For simplicity, we take $h_j = h$, and assume we dealing with the problem on the $(j+1)^{th}$ step. We also denote the solution of (1) with an initial condition y_j at t_j by $y(t; t_j, y_j)$.

3. Interval method for linear ODEs

Let $\hat{y}_{i0}$ in (1) be the middle point of the interval $[\underline{y}_{i0}, \overline{y}_{i0}]$, and consider the following problem

$$\begin{cases} y^{(n)} = \sum_{i=0}^{n-1} p_i(t) y^{(i)} + p_{-1}(t), & t \in [0, r], \\ y^{(i)}(0) = \hat{y}_{i0}, \quad i = 0, 1, \ldots, n-1. \end{cases} \tag{2}$$

Following [16], the solution of (2) can be written as a power series:

$$y(x) = \sum_{k=0}^{\infty} a_k x^k,$$

which converges for all x with $|x| < r$ and for some $\kappa \in \mathbb{N}$ the partial sum $A(\kappa) = \sum_{k=0}^{\kappa-1} a_k x^k$ has been considered as an approximate solution and $t(\kappa) = \sum_{k=\kappa}^{\infty} a_k x^k$ as the truncation error. Indeed after determining a suitable κ and approximate solution $A(\kappa) = \sum_{k=0}^{\kappa-1} a_k x^k$ a tight enclosure like $[-s(\kappa), s(\kappa)]$ has been computed which includes the truncation error $t(\kappa)$.

It is also pointed out in [16], for equation (2), there are guaranteed enclosures $A_i(\kappa)+[-s_i(\kappa), s_i(\kappa)]$, such that $y^{(i)}(h) \in A_i(\kappa)+[-s_i(\kappa), s_i(\kappa)]$, for $i = 0, 1, \ldots, n-1$, where

$$A_i(\kappa) = \sum_{k=i}^{n-1} (k-i+1)_i \frac{\hat{y}_{k0}}{k!} h^{k-i} + \sum_{k=n}^{\kappa-i} a_k h^{k-i}, \tag{3}$$

and

$$a_{k+n} = \sum_{i=0}^{n-1} \sum_{j=0}^{k} \frac{(k-j+1)_i b_{ij}}{(k+1)_n} a_{k+i-j} + \frac{b_{-1,k}}{(k+1)_n}, \quad k = 0, 1, \ldots$$

with $(k_0) = 1$, $(k)_i = k(k+1)\dots(k+i-1)$ for $i, k \in \mathbb{N}$.

In this case the interval $[-s_i(\kappa), s_i(\kappa)]$ followed by the Neher's algorithm [16] is a very tight interval which contains the remainder series $\sum_{k=\kappa}^{\infty} a_k h^{k-i}$.

On the other hand, equation (2) may be written as a system of the following initial value problem:

$$\begin{cases} u_0' = u_1, \\ u_1' = u_2, \\ \quad\vdots \\ u_{n-1}' = \sum_{i=0}^{n-1} p_i(t)u_i + p_{-1}(t), \qquad t \in [0, r], \\ u_i(0) = \hat{y}_{i0}, \qquad\quad i = 0, 1, \dots, n-1. \end{cases} \tag{4}$$

This equation has an interval solution as

$$u(h; t_0, \hat{y}_0) = \{u_0(h; t_0, \hat{y}_0), u_1(h; t_0, \hat{y}_0), \dots, u_{n-1}(h; t_0, \hat{y}_0)\}, \tag{5}$$

such that,

$$u_i(h; t_0, \hat{y}_0) \in A_i(\kappa) + [-s_i(\kappa), s_i(\kappa)], \qquad i = 0, 1, \dots, n-1. \tag{6}$$

In this position, our aim is to use a local coordinate transformation on $u(h, t_0, \hat{y}_0)$, which has been proposed by Moore in [7, 13]. Suppose

$$v_0 = \{v_{00}, v_{10}, \dots, v_{(n-1)0}\} \in Y_0 = \prod_{i=0}^{n-1} [\underline{y}_{i0}, \overline{y}_{i0}],$$

and $\hat{y}_0 = \{\hat{y}_{00}, \hat{y}_{10}, \dots, \hat{y}_{(n-1)0}\}$. Let $u(h; t_0, v_0)$ and $u(h; t_0, \hat{y}_0)$ be the exact solutions of equation (2) at $t = h$, with initial conditions v_0 and $\hat{y}_0$, respectively.

Considering the expansion $u(h; t_0, v_0)$ around $\hat{y}_0$ gives us:

$$u(h; t_0, v_0) = u(h; t_0, \hat{y}_0) + C(h, \hat{y}_0)(v_0 - \hat{y}_0) + O(||v_0 - \hat{y}_0||),$$

where the entries of the matrix $C(h, \hat{y}_0)$ are given by

$$C_{ij}(h, \hat{y}_0) = \frac{\partial u_i(h; t_0, v)}{\partial v_j}|_{v=\hat{y}_0}, \tag{7}$$

such that $v = \{v_0, v_1, \dots, v_{n-1}\}$.

Let us set

$$\tilde{u}(h; t_0, v_0) = u(h; t_0, \hat{y}_0) + C(h, \hat{y}_0)(v_0 - \hat{y}_0),$$

which is an appropriate approximation to $u(h; t_0, v_0)$, when $||v_0 - \hat{y}_0||$ is small, and let $z(h, \hat{y}_0, v_0)$ be defined by:

$$u(h; t_0, v_0) = u(h; t_0, \hat{y}_0) + C(h, \hat{y}_0)z(h, \hat{y}_0, v_0). \tag{8}$$

The function $z(h, \hat{y}_0, v_0)$ is well-defined since $C(h, \hat{y}_0)$ is non-singular. Differentiating from equation (8) with respect to t gives us

$$u'(h; t_0, v_0) = u'(h; t_0, \hat{y}_0) + C'(h, \hat{y}_0)z(h, \hat{y}_0, v_0) + C(h, \hat{y}_0)z'(h, \hat{y}_0, v_0). \tag{9}$$

For simplicity, we consider $z = z(h, \hat{y}_0, v_0)$ and $C(h) = C(h, \hat{y}_0)$. The following interval initial value problem is then obtained from equation (9):

$$\begin{aligned} z' &= C^{-1}(h)(u'(h; t_0, v_0) - u'(h; t_0.\hat{y}_0) - C'(h)z), \\ &= C^{-1}(h)(f(u(h; t_0, v_0)) - f(u(h; t_0, \hat{y}_0)) - C'(h)z), \\ &= C^{-1}(h)(f(u(h; t_0, \hat{y}_0) + c(h)z) - f(u(h; t_0, \hat{y}_0)) - C'(h)z), \end{aligned} \tag{10}$$

where $z(t_0) = v_0 - \hat{y}_0 \in Y_0 - \hat{y}_0$.

This interval initial value problem may be solved at $t = h$, by one of the known interval methods such as Moore's method, polynomial enclosure scheme or constant enclosure method (see e.g. [13] and references therein). Let Z_1 be a solution of this equation. If we substitute Z_1 in equation (8), we obtain an interval Y_1 such that

$$u(h; t_0, v_0) \in Y_1.$$

In this position, assuming that $v_0 \in Y_0$ yields $u(h; t_0, Y_0) \subseteq Y_1$, therefore Y_1 is an interval solution for equation (4). In [7, 13], Moore has been used a more tractable version of the transformation in the equation (8). He has considered some approximations for $u(h; t_0, \hat{y}_0)$ and $C(h, \hat{y}_0)$, in case where they are not available. Note that we do not need to use the exact values for $u(h; t_0, \hat{y}_0)$ and $C(h, \hat{y}_0)$, since we only need to set up a local coordinate system. In the proposed algorithm which will be given in the next section we apply this idea including the Neher's guaranteed enclosures (5) to compute an interval solution for equation (1). To do so, we should obtain $C(h, \hat{y}_0)$, which is so important for the proposed idea and can be approximated by the following theorem:

Theorem 1. *Suppose that $u(h; t_0, v)$ is the solution of equation (4), where $v = \{v_0, v_1, \dots, v_{n-1}\}$ is the initial condition and $C(h, \hat{y}_0)$ is given by (7). Then*

$$C(h, \hat{y}_0) \cong \begin{bmatrix} 1 & h & \frac{h^2}{2} & \dots & \frac{h^{n-1}}{(n-1)!} \\ 0 & 1 & h & \dots & \frac{h^{n-2}}{(n-2)!} \\ 0 & 0 & 1 & \dots & \frac{h^{n-3}}{(n-3)!} \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

Proof. From relations (3), (5) and (6) we conclude

$$u(h;t_0,v)=\begin{cases}\displaystyle\sum_{k=0}^{n-1}\frac{v_k}{k!}h^k+\sum_{k=n}^{\kappa-1}a_kh^k+[-s_0(\kappa),s_0(\kappa)],\\ \displaystyle\sum_{k=1}^{n-1}(k)_1\frac{v_k}{k!}h^{k-1}+\sum_{k=n}^{\kappa-1}(k)_1a_kh^{k-1}+[-s_1(\kappa),s_1(\kappa)],\\ \vdots\\ \displaystyle v_{n-1}+\sum_{k=n}^{\kappa-1}(k-(n-1)+1)_{(n-1)}a_kh^{k-(n-1)}+[-s_{n-1}(\kappa),s_{n-1}(\kappa)].\end{cases}$$

Truncating $u(h;t_0,v)$ from $k=n$ and using (7), for $i=0,1,\dots,n-1$, we obtain

$$c_{ij}(h,\hat{y}_0)\cong\frac{\partial(\sum_{k=i}^{n-1}(k-i+1)_i\frac{v_k}{k!}h^{k-i})}{\partial v_j}\Bigg|_{v=\hat{y}_0}=\begin{cases}\dfrac{h^{j-i}}{(j-i)!}, & j\geq i,\\ 0, & j<i,\end{cases}$$

and this is the result of the theorem. □

Now, let us set $C(h)=C(h,\hat{y}_0)$. We can easily compute $C^{-1}(h)$ and insert the result into (10), which yields an ordinary differential equation with interval initial conditions that is called the Transformed Differential Equation (TDE). This TDE can be solved using some of the well-known interval based methods (e.g. Moore's method) to obtain an interval solution as

$$Z_1=\{Z_{01},Z_{11},\dots,Z_{n-1,1}\}.$$

Using (8), we have

$$u(h;t_0,v_0)\in u(h;t_0,\hat{y}_0)+C(h)Z_1=Y_1,\quad\forall v_0\in Y_0,\tag{11}$$

where $Y_1=\{Y_{01},Y_{11},\dots,Y_{n-1,1}\}$, which yields $y(h)\in Y_{01}$, $y'(h)\in Y_{11},\dots,$ $y^{(n-1)}(h)\in Y_{n-1,1}$.

4. An algorithm for a linear second order ODEs with interval initial conditions

In this section, we construct an algorithm for the case $n=2$, which is a linear second order ODEs with interval initial conditions as

$$\begin{cases}y''=p_0(t)y+p_1(t)y'+p_{-1}(t),\\ y_0\in Y_{00}=[\underline{y}_{00},\overline{y}_{00}],\qquad y'_0\in Y_{10}=[\underline{y}_{10},\overline{y}_{10}].\end{cases}\tag{12}$$

At first, we write the equation as a set of initial value problems

$$\begin{cases} u_0' = u_1, \\ u_1' = p_0(t)u_0 + p_1(t)u_1 + p_{-1}(t), \\ u_0(0) \in U_{00} = [\underline{y}_{00}, \overline{y}_{00}], \\ u_1(0) \in U_{10} = [\underline{y}_{10}, \overline{y}_{10}]. \end{cases}$$

Equation (12) can be solved by Neher's algorithm [16] with $\hat{y}_0 = \{\hat{y}_{00}, \hat{y}_{10}\}$ as initial conditions, to obtain the following interval solution

$$u(h; t_0, \hat{y}_0) = \begin{cases} A_0(\kappa) + [-s_0(\kappa), s_0(\kappa)], \\ A_1(\kappa) + [-s_1(\kappa), s_1(\kappa)]. \end{cases}$$

Now, we can get the following differential equation from (10), which is known as TDE, for $n = 2$

$$\begin{cases} z_1' = -h\{p_0(h)(z_1 + hz_2) + p_1(h)z_2\}, \\ z_2' = p_0(h)(z_1 + hz_2) + p_1(h)z_2, \\ z_1(0) \in Z_{10} = U_{00} - \hat{y}_{00}, \\ z_2(0) \in Z_{20} = U_{10} - \hat{y}_{10}, \end{cases} \tag{13}$$

that can be solved easily by the following interval based method

$$\begin{cases} Z_{11} = Z_{10} + hz_1' = Z_{10} + h\{-h\{p_0(h)(Z_{10} + hZ_{20}) + p_1(h)Z_{20}\}\}, \\ Z_{21} = Z_{20} + hz_2' = Z_{20} + h\{p_0(h)(Z_{10} + hZ_{20}) + p_1(h)Z_{20}\}. \end{cases} \tag{14}$$

Inserting $Z_1 = \{Z_{11}, Z_{21}\}$ and $u(h; t_0, \hat{y}_0)$ in (11) yields $U_1 = \{U_{01}, U_{11}\}$ that is a solution for (12), such that

$$u(h; t_0, \{U_{00}, U_{01}\}) \subseteq u(h; t_0, \hat{y}_0) + C(h)Z_1 = U_1. \tag{15}$$

For the next step, we should solve (12) with U_{01} and U_{11}, as initial conditions to obtain $U_2 = \{U_{02}, U_{12}\}$, such that

$$u(h; t_1, \{U_{01}, U_{11}\}) \subseteq U_2. \tag{16}$$

The following algorithm summarizes our strategy for solving the linear second order ODEs (12):

Algorithm: Interval based method for linear ODEs with analytic coefficients

Step 1. Input:

$h, \quad m, \quad p_{-1}(t), \quad p_i(t), \quad Y_{i0}, \quad i = 0, 1;$

Step 2. For $\quad k = 0, 1, 2, \cdots,$ Do

Step 3. Compute: $\quad \hat{y}_{i0} = m(Y_{i0})$, for $i = 0, 1$.

Step 4. Solve equation (12) by Neher's algorithm [16] with $\hat{y}_{i0}$, $i = 0, 1$, as initial conditions to obtain $u(h; t_k, \hat{y}_0)$.

Step 5. Set $Z_{(i+1)0} = Y_{i0} - \hat{y}_{i0}$ in (14) to arrive at Z_{i1}, for $i = 0, 1$.

Step 6. Substitute $Z_1 = \{Z_{11}, Z_{21}\}$ and $u(h; t_k, \hat{y}_0)$ in (15) to obtain $U_1 = \{U_{01}, U_{11}\}$.

Step 7. Set $Y_{i0} = U_{i1}$ and go to step 3.

End-For

Step 8. Output:

$$Y_m = \{Y_{0m}, Y_{1m}, \ldots, Y_{n-1,m}\}.$$

The accuracy of the algorithm depends on the width of the initial conditions, the step size h, and the type of interval method that have been used to solve the TDE. In addition, by increasing n, the accuracy will be increased (see e.g. the proof of theorem (1)). We used an interval based method (14) to solve the equation (13). However, the other improved interval methods such as the Hermite-Obreschkoff method [10] can be used iteratively on (13), to obtain a reasonable solution that makes decreasing the wrapping effect. Note that, the accuracy in the Neher's algorithm is controlled by adjusting two parameters ϵ_{rel}, $\epsilon_{abs} > 0$ (see e.g. [16]), so if we decrease the values of ϵ_{rel} , ϵ_{abs}, more accurate solution will be achieved. Subsequently, the algorithm leads to more accurate results.

We emphasize that, the step 5 of the proposed algorithm may causes some overestimations and wrapping effects in the output interval solution. For example, very large coefficients in $p_i(x)$, $i = 1, \ldots, n-1$ may produce a big overestimation in the given time step, which will be transferred to the next step and causes wrapping effect. In such case, we should use smaller step size h with large number of iteration that again causes wrapping effect and finally the algorithm breaks down. However, various effective interval based methods such as the Hermite-Obreschkoff method [10], may be applied for step 5 to decrease overestimations and prevent the algorithm from breaking down.

Some advantages of the introduced method may be summarize as follows:

• It is well-known that the Neher's algorithm [16] is very accurate, and any method based on this algorithm will be accurate and effective as well. Therefore, employing this algorithm, will lead to a reasonable solution for our main problem.

• In this method, only one step is needed to solve the TDE which spares the results from the wrapping effect. In addition, we can use the Hermit - Obreschkoff method ([10]) repeatedly, to arrive at the desired solution for the TDE.

• The TDE has symmetric initial conditions which, decreases the overestimation in the interval computations (see [13, 8, 9] for further details).

• The algorithms AWA [11], VNODE [11] and some others need a priori enclosure of the solution to start the computation. Usually, a priori enclosure can only be secured for small step sizes. Therefore choosing the suitable step size h is a restriction for these methods, while our proposed method is not. This restriction may be seen in Tables 1-3 in the next section.

5. Numerical experiments and some comments

Here we consider three test problems and report the numerical results by the proposed method. Our numerical results can be compared with those obtained by AWA method. All computations are performed by using the software Mathematica®.

For computational purpose, we continue with details and clarify that how the solution at $t = 0.1$ is obtained for the first example. All the examples have been solved at $t = 1$ with different step sizes by the proposed method and compared with the results of AWA method.

Example 1.

$$\begin{cases} y'' = e^t y + e^{-t} - 1, \\ y(0) \in [0.99999, 1.00001], \\ y'(0) \in [-1.00001, -0.99999]. \end{cases}$$

Following the algorithm, we should solve the problem with initial conditions $\hat{y}_{00} = 1$ *and* $\hat{y}_{10} = -1$ *by the Neher's algorithm such that if we set* $\kappa = 10$, *and* $h = 0.1$, *we obtain* $A_0(\kappa) = 0.904837$, $s_0(\kappa) = 1.11022 \times 10^{-16}$, *and* $A_1(\kappa) = -0.904837$, $s_1(\kappa) = 1.11022 \times 10^{-16}$. *So we have*

$$u(0.1, 0, \hat{y}_0) = \begin{cases} 0.904837 + [-1.11022 \times 10^{-16}, 1.11022 \times 10^{-16}], \\ -0.904837 + [-1.11022 \times 10^{-16}, 1.11022 \times 10^{-16}]. \end{cases}$$

In this position, we solve the following interval initial value problem:

$$\begin{cases} z_1' = -he^h(z_1 + hz_2), \\ z_2' = e^h(z_1 + hz_2), \\ z_{10} \in [-0.00001, 0.00001], \\ z_{20} \in [-0.00001, 0.00001]. \end{cases}$$

For solving the above IVP, from (14) we obtain

$$Z_{11} = [-0.0000101216, 0.0000101216] \text{ and } Z_{21} = [-0.0000112157, 0.0000112157].$$

Now, we use (16) to arrive at

$$U_{01} = [0.904826, 0.904848] \text{ and } U_{11} = [-0.904849, -0.904826].$$

By the proposed method, the interval solution at $t = 0.1$ *is* $Y_{01} = [0.904826, 0.904848]$, *meanwhile the exact interval solution is* $[0.904827, 0.904847]$. *However, the solution* $[0.899989, 1.00001]$ *will be obtained by the AWA method for the same value of* h.

The exact interval solution at $t = 1$ *is also* $[0.367872, 0.367887]$. *The example has solved at* $t = 1$ *for different values of* h, *by the proposed method and AWA. We reported the results in Table 1 which compares the exact and the approximate solutions obtained by both methods.*

As can be seen, the results obtained by the proposed method are more accurate than those obtained by AWA method.

h	Exact sol. [0.367872, 0.367887] Our method	Exact sol. [0.367872, 0.367887] AWA method
0.0001	[0.367872 0.367887]	[0.367838 0.367893]
0.001	[0.367852 0.367907]	[0.367523 0.367959]
0.005	[0.367852 0.367907]	[0.366249 0.368169]
0.01	[0.367852 0.367907]	[0.364689 0.368491]
0.1	[0.367847 0.367912]	[0.342457 0.383423]
0.2	[0.367841 0.367918]	[0.327996 0.412606]
0.5	[0.367809 0.367950]	[0.313789 0.500015]
1	[0.367751 0.368008]	[0.303591 1.535210]

Table 1: *The results obtained for Example 1 with the proposed and AWA methods at* $t = 1$

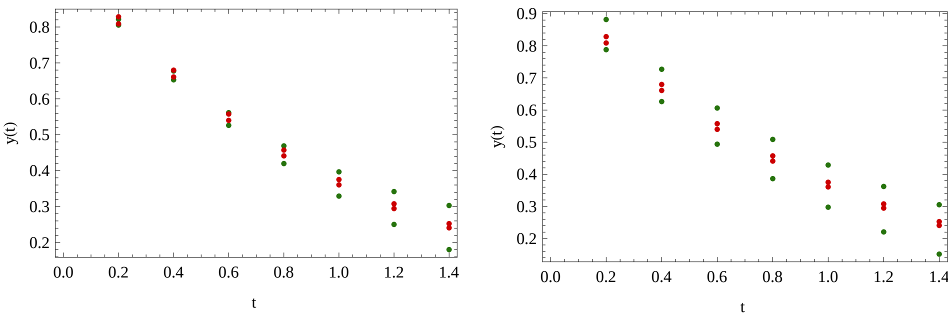

Figure 1: *The results obtained for Example 1 by the proposed method (left, green), AWA method (right, green) and exact bounds (red)*

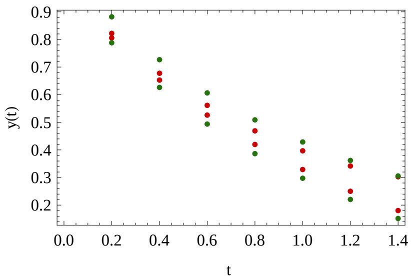

Figure 2: *The results obtained for Example 1, by the proposed method (red dots) and AWA method (green dots)*

Example 2.

$$\begin{cases} y'' = y, \\ y(0) \in [0.99999, 1.00001], \\ y'(0) \in [-1.00001, -0.99999], \end{cases}$$

with the exact interval solution $[0.367872, 0.367887]$ *at* $t = 1$*. We solved this problem by using AWA and the proposed algorithm at* $t = 1$ *and reported the results in*

Table 2. Due to the simple form of the problem we also achieved good results by the AWA method. Figures 3-4 represent the results obtained from AWA and the proposed algorithm with the initial conditions $y(0) \in [0.99, 1.01]$, $y'(0) \in [-1.01, -0.99]$ and $h = 0.2$. In this case, the results of both methods are almost similar.

h	Exact sol. [0.367872, 0.367887] Our method	AWA method
0.0001	[0.367850 0.367909]	[0.367852 0.367907]
0.001	[0.367852 0.367907]	[0.367852 0.367907]
0.005	[0.367852 0.367907]	[0.367852 0.367907]
0.01	[0.367852 0.367907]	[0.367852 0.367907]
0.1	[0.367849 0.367910]	[0.367772 0.368022]
0.2	[0.367844 0.367915]	[0.367230 0.368841]
0.5	[0.367824 0.367935]	[0.359565 0.383165]
1	[0.367739 0.368019]	[-0.444527 0.111199]

Table 2: *The results obtained for Example 2 with the proposed and AWA methods at $t = 1$*

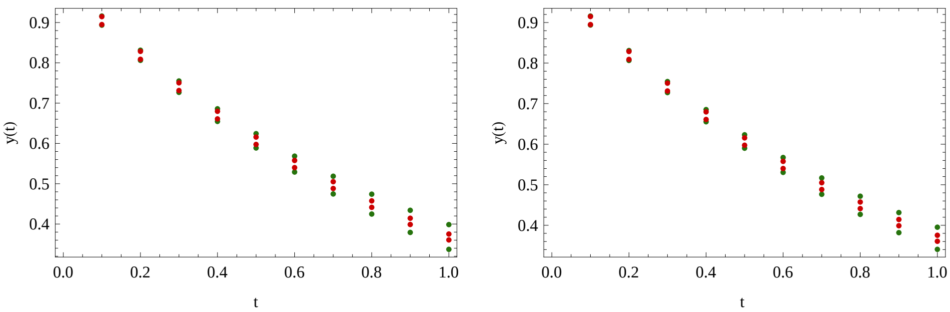

Figure 3: *The results obtained for Example 2 by the proposed method (left, green), AWA method (right, green) and exact bounds (red)*

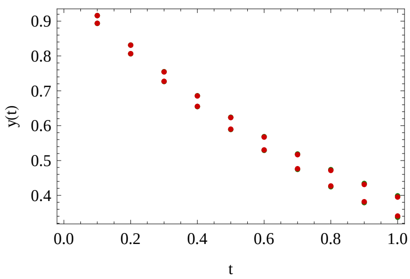

Figure 4: *The results obtained for Example 2, by the proposed method (red dots) and AWA method (green dots)*

Example 3.

$$\begin{cases} y^{(4)} = (t^2+10t+26)y''' + (-20t-99.5)y'' + (t^2+10t+25)y' + (-2t^2-4t+29.5)y, \\ y(0) \in [4.9999, 5.0001], \quad y'(0) \in [3.9999, 4.0001], \\ y''(0) \in [2.9999, 3.0001], \quad y'''(0) \in [1.9999, 2.0001], \end{cases}$$

The exact solution of this IVP, with $y(0) = 5, y'(0) = 4, y''(0) = 3, y'''(0) = 2$ *is expressed as* $y(t) = (5-t)e^t$. *The exact interval solution of the problem at* $t = 1$ *is* $[10.8731, 10.8733]$. *We have reported the results of the proposed method and AWA in Table 3. Figures 5-6 represent numerical results obtained by both methods with initial conditions* $y(0) \in [4.9, 5.1], y'(0) \in [3.9, 4.1], y''(0) \in [2.9, 3.1], y'''(0) \in [1.9, 2.1]$ *and* $h = 0.01$.

	Exact sol. [10.8731, 10.8733]	
h	Our method	AWA method
0.01	[10.8727 10.8736]	[10.8712 10.8739]
0.1	[10.8716 10.8747]	[10.0841 10.9002]
0.2	[10.8704 10.8759]	[9.34705 10.7785]
0.5	[10.8685 10.8777]	[7.41898 11.4971]
1	[10.8704 10.8758]	[4.99990 13.1552]

Table 3: *The results obtained for Example 3 with the proposed and AWA methods at* $t = 1$

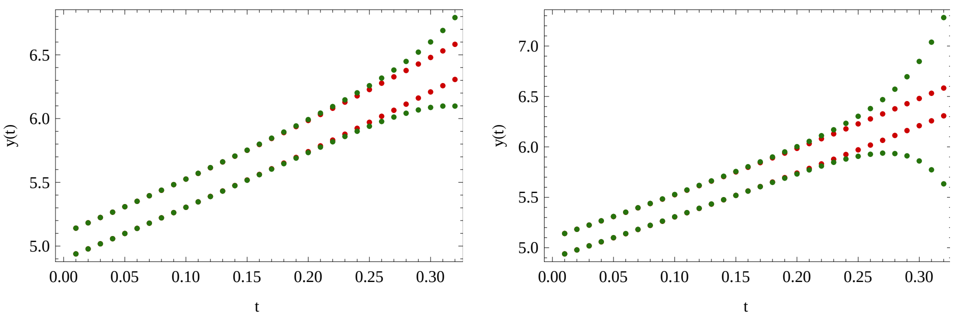

Figure 5: *The results obtained for Example 3 by the proposed method (left, green dots) and AWA method (right, green dots) and the exact bounds (red dots)*

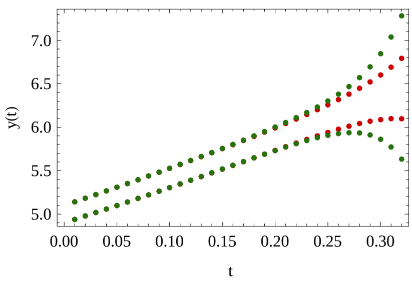

Figure 6: *The results obtained for Example 3, by the proposed method (red dots) and AWA method (green dots)*

6. Conclusion

The n^{th} order linear ODEs have many applications in science and engineering, but in many cases, accumulations of truncation and round off errors make restriction for the convergence of the numerical methods to the exact solution. In [16], Neher has suggested a verified algorithm which computes a guaranteed bound for the solution including all truncation and round off errors. However, in this algorithm interval initial conditions are not considered and these make some restriction, i.e. the algorithm may not continue for more than one step. In this paper we have presented a new algorithm, which determines a validated solution for an n^{th} order linear ODE with interval-valued initial conditions. We have shown that, the proposed algorithm employs the well known accurate Neher's algorithm (step 4 in the algorithm), therefore it is effective and accurate. Also, unlike the existing interval methods, the new approach does not need any priori enclosure and the step size is not restricted to small values in giving accurate results. In this method, might be some overestimations and wrapping effects; however, it can be decreased by choosing an appropriate interval method (step 5 in the algorithm). Numerical examples show that this approach is an effective scheme to obtain tight enclosures of the solution to the n^{th} order linear ODEs.

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