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An approach based on matrix polynomials for linear systems of partial differential equations

Abstract

In this paper, an approach based on matrix polynomials is introduced for solving linear systems of partial differential equations. The main feature of the proposed method is the computation of the Smith canonical form of the assigned matrix polynomial to the linear system of PDEs, which leads to a reduced system. It will be shown that the reduced one is an independent system of PDEs having only one unknown in each equation. A comparison of the results for several test problems reveals that the method is very effective and convenient. The basic idea described in this paper can be employed to solve other linear functional systems.

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1. Introduction

Systems of partial differential equations arise in many scientific fields such as solid state physics, fluid dynamics, mathematical biology and chemical kinetics [8, 11, 17]. There are some well-known numeric-analytic methods such as Exp-function method [8], the Adomian decomposition method [17], the variational iteration method [14], and the differential transform method [3] for obtaining the closed form approximate solution of linear systems of PDEs. Most of these techniques are based on the substitution of an initial (nontrivial) approximate solution in the corresponding partial differential equation, and they mainely depend on the choice of the auxiliary operators. Moreover, by increasing the dimension of the system, these methods usually encounters difficulties with high computational complexity. However, finding accurate and efficient methods for solving linear systems of PDEs has long been an active research undertaking.
On the other hand, recently, a great deal of attention has been devoted to the study of matrix polynomials, which was extensively studied by Gohberg et al. [7], with particular focus on its applications. In this work, we present an alternative new approach based on the Smith decomposition form of the matrix polynomials for finding solutions of linear systems of PDEs. The symbolic computation of Smith canonical form of matrix polynomials is also a widely studied topic, which is well addressed in [18]. This method was successfully applied to system of operator equations [15], linear functional systems [1], system of ordinary differential equations [7, 9, 12], system of integral equations [16] and linear differential algebraic equations [2].
The principle task in this paper, is to introduce a novel method for obtaining the exact solution of linear system of PDEs. First, a matrix polynomial equation is assigned to the system, and the explicit system is presented by considering its corresponding matrix polynomial equation. Then, an equivalent system, consisting of linear equations having only one

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unknown in each equation, is obtained by applying the Smith form of the mentioned matrix polynomial. In favorable cases, (see for instance, the problems studied in Section 3) it may even turn out that the obtained independent system can be evaluated analytically.
This paper is organized as follows: In Section 2, a matrix polynomial based method has been introduced. The efficiency and accuracy of the method are demonstrated through several test problems in Section 3 and advantages of the method compared with other numeric-analytic methods are discussed. Finally, the conclusion appears in Section 4.

2. The basic idea

In this section, we sketch the proposed method based on matrix polynomials for computing solutions of linear systems of partial differential equations.

2.1. Matrix polynomials

Here, we recollect some basic definitions and properties of matrix polynomials that are needed in the sequel. For more details, we refer the reader to [6, 7].
Recall that a matrix polynomial over $\mathbb{R}[\lambda]$ is simply a matrix with polynomial elements in λ with coefficients in $\mathbb{R}$ which can alternatively be considered as a polynomial with matrix coefficients. Such matrices arise in the mathematical treatment of various types of linear systems. (See for instance [7, 13]).

Definition 2.1.
A square matrix polynomial is called unimodular if its determinant is a nonzero constant.

Theorem 2.2.
(From [7]) *Let $A(\lambda)$ be an $n \times n$ full rank matrix over $\mathbb{R}[\lambda]$. There exist unimodular $n \times n$ matrix polynomials $P(\lambda)$ and $Q(\lambda)$ such that*

$$P(\lambda)A(\lambda)Q(\lambda) = D(\lambda), \tag{2.1}$$

where $D(\lambda) = diag[d_1(\lambda), \cdots, d_n(\lambda)]$ is called the **Smith canonical form** *of $A(\lambda)$, with monic scalar polynomials $d_i(\lambda)$ such that $d_i(\lambda)$ is divisible by $d_{i-1}(\lambda)$.*

The following lemma develops the basis of our method. It establishes the correspondence between the solution of a general system of operator equations and an independent system, which has been pointed out in [7] for system of ODEs.

Lemma 2.3.
(From [7]) *Let $D(\lambda)$ be the Smith canonical form of the matrix $A(\lambda)$ as (2.1). Then the system given in the form $A(\lambda)X = G$, can be reduced to the system*

$$D(\lambda)Y = P(\lambda)G, \tag{2.2}$$

while the unknown vector X being known from:

$$X = Q(\lambda)Y. \tag{2.3}$$

It is a marked advantage of the method that the reduced structure (2.2) include one unknown in each equation, since $D(\lambda)$ is a diagonal matrix.

2.2. Analysis of the method

In the sequel, we give a slightly modified version of the proposed method in Lemma 2.3 for obtaining the solution of a linear system of PDEs in a more efficient way. We proceed as follows:
To clarify the basic ideas, consider the scalar unknown function $u_i(x_1, \cdots, x_k)$, being a particular solution of the partial differential system:

$$L_i(u_1, u_2, \cdots, u_n) = g_i, \qquad i = 1, \cdots, n, \tag{2.4}$$

with convenient initial data in a compact domain Ω of that space, such that L_i is a linear operator of a certain order N:

$$L_i = \sum_{m=m_0}^{N} \sum_{|J|=m} a_J D_J^m,$$

where $m_0 \geq 0$, is the minimal order, $a_J \in \mathbb{R}$, $J = (j_1, \cdots, j_k) \in \mathbb{N}^k$, $|J| = \sum_{n=1}^{k} j_n$, and $D_J^m = \partial^m / \partial x_1^{j_1} \cdots \partial x_k^{j_k}$. In addition, g_i are source terms in a k dimensional space. We suppose that the smoothness assumptions, to be made, will ensure that the differential operators L_i are completely continuous acting on functions.
Considering the linear operator L_i, the system (2.4) can be written as

$$\sum_{j=1}^{n} L_{ij} u_j = g_i, \qquad i = 1, \cdots, n, \tag{2.5}$$

where $L_{ij} := L_i(u_j)$. To proceed in depth, let us consider the coupled system for the case $k = 2$, so $u_i = u_i(x, t)$, would be the solution of (2.5). (The study of the system for higher size remains a challenge). Suppose that there exists two linear differential operators

$$\lambda = \frac{\partial}{\partial t}, \qquad \mu = \frac{\partial}{\partial x}, \tag{2.6}$$

such that (2.5) can be turned into a matrix polynomial equation as follows:

$$A(\mu)[\lambda] X = G, \tag{2.7}$$

where $A(\mu)[\lambda] = [L_{ij}(\mu, \lambda)]_{n\times n}$ is a bivariate matrix polynomial, $G = [g_i(x, t)]_{n\times 1}$ is the source term, and $X = [u_i(x, t)]_{n\times 1}$ is the unknown vector should be determined.
One obvious approach to study bivariate matrix polynomials is the extension of the well known results for univariate matrix polynomials, such that the principal variable of the bivariate matrix polynomial is considered as the variable that emerges in practice among all of the variables, and other variables are considered as constants. More precisely, when dealing with matrices in $\mathbb{R}[\lambda, \mu]$, it is necessary to be clear which ring is being used, $\mathbb{R}(\mu)[\lambda]$ or $\mathbb{R}(\lambda)[\mu]$. Throughout this paper, we concentrate on $\mathbb{R}(\mu)[\lambda]$, the ring of polynomials in λ in which its coefficients are functions in term of the constant μ. For the problem of reducing a bivariate matrix polynomial to Smith canonical form see e.g. [4, 5].
In order to allow the applicability of the proposed method, first we should obtain the Smith canonical form (2.1) in $\mathbb{R}(\mu)[\lambda]$, while the matrices $P(\mu)[\lambda]$ and $Q(\mu)[\lambda]$ should be unimodular. Following the same arguments, the reduced system is

$$D(\mu)[\lambda] Y = P(\mu)[\lambda] G, \tag{2.8}$$

which contains n independent equations in Y, and the unknown vector is constructed with

$$X = Q(\mu)[\lambda] Y. \tag{2.9}$$

The following algorithm summarizes the proposed matrix polynomial approach for the problem:

Algorithm: Matrix polynomial approach for linear system of PDEs

Consider the linear system of PDEs (2.5) for $n = 2$
Step 1. Set the variables of the matrix polynomial with two linear differential operators $\lambda = \frac{\partial}{\partial t}$, $\mu = \frac{\partial}{\partial x}$.
Step 2. Turn the linear system of PDEs to a matrix polynomial equation like (2.7).
Step 3. Construct the Smith canonical form in $\mathbb{R}(\mu)[\lambda]$, as $P(\mu)[\lambda]A(\mu)[\lambda]Q(\mu)[\lambda] = D(\mu)[\lambda]$.
Step 4. Develop the reduced system as (2.8).
Step 5. Solve (2.9) for the unknown vector.

We emphasize that although there exist other symbolic procedures to solve the systems of partial differential equations, here our aim is to obtain a framework based on matrix polynomials which may be applied for other general systems. Noting that, the important concept in this reduction is equivalence of the main system and a system of independent equations. Exploiting the analytical results of independent equations facilitates the solvability of the reduced system.

3. Illustrative examples

In this section, the exact solution of various class of linear systems of partial differential equations (homogenous, nonhomogenous and higher order PDEs) is presented to assess the efficiency of the method. All the computations associated with the examples were performed using computer algebra package Maple®.

Example 3.1 (from [14]).
Consider the following homogenous linear system of partial differential equations

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial v}{\partial x} + (u+v) = 0, \\ \dfrac{\partial v}{\partial t} - \dfrac{\partial u}{\partial x} + (u+v) = 0, \end{cases}$$

with initial conditions $u(x,0) = \sinh(x)$, $v(x,0) = \cosh(x)$, and the exact solution $u(x,t) = \sinh(x-t)$, $v(x,t) = \cosh(x-t)$. Selecting the operators in (2.6), the system is transformed to (2.7) with $G = 0$, $X = [u, v]^T$ and

$$A(\mu)[\lambda] = \begin{pmatrix} 1+\lambda & 1-\mu \\ 1-\mu & 1+\lambda \end{pmatrix}.$$

Now, the Smith canonical form with respect to λ, equivalently in $\mathbb{R}(\mu)[\lambda]$, is as follows:

$$P = \begin{pmatrix} \frac{-1}{\mu-1} & 0 \\ \lambda+1 & \mu-1 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & \lambda^2+2\lambda-\mu^2+2\mu \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 1 \\ 1 & \frac{\lambda+1}{\mu-1} \end{pmatrix}.$$

Note that P and Q are unimodular for all values of λ and $\mu \neq 1$. So we can construct the reduced system according to (2.8) as

$$y_1 = 0, \qquad (\lambda^2 + 2\lambda - \mu^2 + 2\mu)y_2 = 0, \tag{3.1}$$

and the solution (2.9) in the operator form is:

$$u = y_2, \qquad v = y_1 + \frac{\lambda+1}{\mu-1} y_2. \tag{3.2}$$

Substitution of u from (3.2) in (3.1), leads to the following independent equation in u:

$$\left(\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial t}\right) - \left(\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x}\right) = 0.$$

We should mention the diversity of the well-known numeric-analytic methods for solving such an equation, however we approximate the solution with spectral type method, considering $u(x,0) = sinh(x)$. Consequently, multiplying both sides of the second equation of (3.2) by the nonzero $(\mu-1)$, and substituting y_1 and y_2 leads to the independent equation

$$\frac{\partial v(x,t)}{\partial x} - v(x,t) = \sinh(x-t) - \cosh(t-x),$$

for determining the approximate solution of $v(x,t)$, considering $v(x,0) = \cosh(x)$. This system has been solved in [14] with variational iteration method. The computational results reported in Table 1, with 8 number of iterations for both methods, show the accuracy of the method in comparison to the method presented in [14].
The presented scheme is also applicable to nonhomogenous systems of linear PDEs which will be considered in the following:

Table 1. The absolute errors of our method in comparison to the method presented in [14].

		$u(x,t)$		$v(x,t)$	
x	t	Our method	Method in [14]	Our method	Method in [14]
0	0	0	0	0	0
	0.2	1.411447 E-12	2.666666 E-6	6.352029 E-11	2.822720 E-14
	0.4	7.234504 E-10	8.533334 E-5	1.628289 E-8	2.893099 E-11
	0.6	2.786252 E-8	6.479999 E-4	4.182422 E-7	1.670839 E-9
	0.8	3.720288 E-7	2.730664 E-3	4.190749 E-6	2.973341 E-8
	1.0	2.780945 E-6	8.333308 E-3	2.507925 E-5	2.776723 E-7
0.2	0	2.000000 E-20	0	0	0
	0.2	1.134915 E-11	2.720178 E-6	6.451076 E-11	5.368960 E-7
	0.4	2.540366 E-9	8.704570 E-5	1.646398 E-8	1.718070 E-5
	0.6	5.578558 E-8	6.610035 E-4	4.210253 E-7	1.304674 E-4
	0.8	4.642544 E-7	2.785466 E-3	4.199941 E-6	5.498114 E-4
	1.0	2.212607 E-6	8.500586 E-3	2.502261 E-5	1.678078 E-3
0.4	0	1.000000 E-20	0	0	0
	0.2	2.456522 E-11	2.882859 E-6	6.809027 E-11	1.095339 E-6
	0.4	5.906136 E-9	9.225151 E-5	1.730583 E-8	3.505089 E-5
	0.6	1.416725 E-7	7.005354 E-4	4.407055 E-7	2.661692 E-4
	0.8	1.319169 E-6	2.952058 E-3	4.377691 E-6	1.121658 E-3
	1.0	7.294961 E-6	9.009023 E-3	2.597021 E-5	3.423225 E-3
0.6	0	1.000000 E-20	0	0	0
	0.2	3.876719 E-11	3.161240 E-6	7.440249 E-11	1.697742 E-6
	0.4	9.508940 E-9	1.011597 E-4	1.884222 E-8	5.432780 E-5
	0.6	2.332453 E-7	7.681824 E-4	4.780728 E-7	4.125534 E-4
	0.8	2.227028 E-6	3.237126 E-3	4.731133 E-6	1.738522 E-3
	1.0	1.267008 E-5	9.879023 E-3	2.796009 E-5	5.305759 E-3
0.8	0	0	0	0	0
	0.2	5.452503 E-11	3.566493 E-6	8.370074 E-11	2.368282 E-6
	0.4	1.349337 E-8	1.141278 E-4	2.113481 E-8	7.578508 E-5
	0.6	3.341791 E-7	8.666592 E-4	5.346269 E-7	5.754948 E-4
	0.8	3.224265 E-6	3.652112 E-3	5.274453 E-6	2.425159 E-3
	1.0	1.855370 E-5	1.114550 E-2	3.107210 E-5	7.401232 E-3
1.0	0	1.000000 E-19	0	1.000000 E-19	0
	0.2	7.247114 E-11	4.114881 E-6	9.635819 E-11	3.133869 E-6
	0.4	1.801934 E-8	1.316762 E-4	2.427562 E-8	1.002838 E-4
	0.6	4.485246 E-7	9.999180 E-4	6.126374 E-7	7.615328 E-4
	0.8	4.350902 E-6	4.213670 E-3	6.029455 E-6	3.209126 E-3
	1.0	2.518195 E-5	1.285929 E-2	3.543114 E-5	9.793742 E-3

Example 3.2.

Let us consider the linear nonhomogenous system from [3]:

$$\begin{cases} \dfrac{\partial z}{\partial t} + \dfrac{\partial w}{\partial x} = xe^{t} + e^{x+t}, \\[2ex] \dfrac{\partial z}{\partial x} - \dfrac{\partial w}{\partial t} = e^{t} - e^{x+t}, \end{cases}$$

with the initial data $z(x,0) = x,\ w(x,0) = e^{x}$, and exact solution $z(x,t) = xe^{t},\ w(x,t) = e^{x+t}$.

Defining the same linear operators as (2.6), we can rewrite the system as (2.7) with $X = [z, w]^T$,

$$G = \begin{pmatrix} xe^t + e^{x+t} \\ e^t - e^{x+t} \end{pmatrix}, \qquad A(\mu)[\lambda] = \begin{pmatrix} \lambda & \mu \\ \mu & -\lambda \end{pmatrix}.$$

Neglecting the case of $\mu = 0$, the Smith form over $\mathbb{R}(\mu)[\lambda]$ is:

$$P = \begin{pmatrix} \frac{1}{\mu} & 0 \\ \lambda & \mu \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & \mu^2 + \lambda^2 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 1 \\ 1 & \frac{-\lambda}{\mu} \end{pmatrix}.$$

The second equation of the reduced system (2.8) leads to the independent equation:

$$(\mu^2 + \lambda^2)y_2 = \lambda(g_1) + \mu(g_2),$$

and considering the first equation of (2.9), we have $z = y_2$. So the function z is obtained via the independent equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = xe^t,$$

together with the condition $z(x, 0) = x$, leads to the exact solution, and w is obtained by replacing z in the first equation as:

$$\frac{\partial w}{\partial x} = e^{x+t},$$

which can be solved with direct integration, considering the condition $w(x, 0) = e^x$.

It is obvious that two above independent equations for determining z and w with analytical methods are superior to the numeric-analytic methods for solving the system. Furthermore, the generality of the method to the higher order linear system of PDEs can be considered in the following example:

Example 3.3.
Consider the second order linear nonhomogenous system of PDEs [3]:

$$\begin{cases} \dfrac{\partial^2 z(x,t)}{\partial t^2} - \dfrac{\partial w(x,t)}{\partial x} = 2x^2 - e^t, \\ \dfrac{\partial w(x,t)}{\partial t} + \dfrac{\partial^2 z(x,t)}{\partial x^2} = 2t^2 + xe^t, \end{cases} \tag{3.3}$$

subject to

$$z(x, 0) = 0, \quad z_t(x, 0) = 0, \quad w(x, 0) = x. \tag{3.4}$$

It can be seen that for suitable chosen differential operators (2.6), the system can be reformulated as (2.7) with $X = [z, w]^T$,

$$G = \begin{pmatrix} 2x^2 - e^t \\ 2t^2 + xe^t \end{pmatrix}, \qquad A(\mu)[\lambda] = \begin{pmatrix} \lambda^2 & -\mu \\ \mu^2 & \lambda \end{pmatrix}.$$

Let us suppose $\mu \neq 0$, the Smith form with respect to λ is:

$$P = \begin{pmatrix} -\frac{1}{\mu} & 0 \\ \lambda & \mu \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & \lambda^3 + \mu^3 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 1 \\ 1 & \frac{\lambda^2}{\mu} \end{pmatrix}. \tag{3.5}$$

Following the same argument, the second equation in the reduced system (2.8) is:

$$(\lambda^3 + \mu^3)y_2 = \lambda(g_1) + \mu(g_2),$$

and according to the first equation of (2.9), $z = y_2$, and the operator equation $(\mu^3 + \lambda^3)z = \lambda(g_1) + \mu(g_2)$, leads the function z to be obtained analytically from

$$\frac{\partial^3 z}{\partial x^3} + \frac{\partial^3 z}{\partial t^3} = 0,$$

and w is obtained by replacing z in the first equation of (3.3), as the simple equation:

$$\frac{\partial w(x,t)}{\partial x} = e^t.$$

The two above equations, considering the conditions (3.4), give the exact solution for z and w. We should mention the simplicity of analytical solving of these reduced equations for obtaining the exact solution $z(x,t) = x^2t^2, w(x,t) = xe^t$, instead of solving the main system.

The most notable advantage of the proposed method is analytic solvability of the independent equations rather than system of equations. Some other beneficial points of the method include its generalization for solving higher order problems, the fast solution evaluating, and dealing with a few number of parameters. Another advantage of this novel method will be apparent when it is applied to solve the problems that have no analytical solutions or their analytical solutions can not be determined directly. It is demonstrated that matrix polynomial theory is a promising tool for more complicated linear, homogeneous or nonhomogeneous systems of PDEs.

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