Numer Algor
DOI 10.1007/s11075-013-9729-4

ORIGINAL PAPER

Numerical analysis of a high order method for state-dependent delay integral equations

M. Khasi · F. Ghoreishi · M. Hadizadeh

Received: 2 May 2012 / Accepted: 5 June 2013

Abstract In this paper we study the piecewise collocation method for a class of functional integral equations with state-dependent delays that is, where the delays depend on the solution. It is well known that these equations typically have discontinuity in the solution or its derivatives at the initial point of integration domain. This discontinuity propagated along the integration interval giving rise to subsequent points, called "singular points", which can not be determined priori and the solution derivatives in these points are smoothed out along the interval. Most of the known numerical methods for this type of equations are generally very sensitive to the singular points and therefore must have a process that detects these points and insert them into the mesh to guarantee the required accuracy. Here, we present a numerical algorithm based on the piecewise collocation method and an approach for tracking the singular points relying on the recent strategy for implicit delay differential equations which has been proposed by Guglielmi and Hairer in 2008. The convergence analysis of the method is investigated and some numerical experiments are presented for clarifying the robustness of the method.

Keywords Delay integral equation · State-dependent delay · Piecewise collocation method · Singular point · Error analysis

Mathematics Subject Classification (2010) 65R20 · 45L05

M. Khasi · F. Ghoreishi · M. Hadizadeh (✉)
Department of Mathematics, K. N. Toosi University of Technology, Tehran, Iran
e-mail: hadizadeh@kntu.ac.ir

M. Khasi
e-mail: khasi@kntu.ac.ir

F. Ghoreishi
e-mail: ghoreishif@kntu.ac.ir

Published online: 29 June 2013

1 Introduction

Introducing the state-dependent delay in modeling real phenomena results frequently from an attempt to account for the influence of some facts on the behavior of a population. Over the past several years it has became apparent that equations with state-dependent delays especially differential type arise in several areas such as in electrodynamics [10], population models [20], economics [3], and engineering [22, 26]. The literature devoted to this subject is concerned fundamentally with the analyzing the state-dependent delay differential equations (**SDDDEs**). The papers by Feldstein et al. [11, 12], Baker et al. [1], Hauber [18], Hartung et al. [17–19], Guglielmi and Hairer [13, 14], Zivaripiran and Enright [28] and the monographs by Bellen and Zennaro [4] and Brunner [6, 7] reflect the development and the current state of the numerical analysis of SDDDEs.

Mathematical models with state-dependent delay integral equations (**SDDIEs**) are wide-spread in applications. For instance, a number of structured population models and models in epidemiology and immunology are specific cases of SDDIEs. However, there are only few works available in the literature for numerical solution of SDDIEs. A family of numerical schemes based on quadrature methods for solving some classes of SDDIEs whose delay appears in the integrand has been developed in Cahlon and Nachman [8], and Cahlon [9]. There are also some works deal with threshold state-dependent delay functional equations in which the delay is implicitly defined through a threshold condition ([19, 20]). Although some important theoretical and numerical results were established in this field, a number of open questions still exist due to the considerable computational complexity compared to the case of constant delays.

Here, we shall consider the following SDDIE:

$$\begin{cases} y(t) = g(t) + (w_\theta y)(t), & t \in I := [a, T], \\ \theta(t, y(t)) \le t, & \\ y(t) = \phi(t), & t \in [\lambda, a), \end{cases} \tag{1.1}$$

where $\lambda = \inf\{\theta(t, y(t)),\ t \in I\}$ and the Volterra integral operator $w_\theta : C[\lambda, T] \to C[\lambda, T]$ is defined as:

$$(w_\theta y)(t) := \int_{\theta(t,y(t))}^{t} k(t, s)y(s)ds.$$

Let the given functions $g(t)$, $k(t, s)$ and delay function $\theta(., .)$ be sufficiently smooth on their respective domains, $\phi(t)$ which is known as initial function is a continuously differentiable function and $y(t)$ is a real valued bounded function to be determined.

The theory of (1.1) appeared from early attempts by many researchers such as Bélair [2] for investigating some population models. The existence and uniqueness results and the asymptotic behavior of the solutions under the differentiability condition for some classes of SDDIEs have also discussed in [2]. To make the problem well defined, a unique solution $y(t)$ is usually identified by specifying an initial function

$\phi(t)$ and some assumptions on the given functions. Therefore, throughout the paper, we assume that the conditions on the given functions g, k, θ and $\phi(t)$ are somehow that the (1.1) possess a unique solution.

These equations typically have discontinuity in the solution or its derivatives at the initial point $t = a$, which may propagate by means of delay function along the integration interval giving rise to subsequent points, called "singular points". More precisely, we observe that $y(a^+) = g(a^+) + (w_\theta y)(a^+)$ may not equal to $y(a^-) = \phi(a)$, even in the case $y(a^+) = y(a^-)$ the derivative of y may be discontinuous at $t = a$. Following [11], these singular points are the odd multiplicity zeroes of the nonlinear equation $\theta(t, y(t)) = Z$, where Z is either a, or any previous point of derivative discontinuity. This ensures us, the delay function goes through the previous jump singular point. Thus, all the singular points obtain from the initial discontinuity at $t = a$. Due to this interdependence, calculation of these points is very difficult, because they should be computed accurately enough to develop high order accuracy. Nevertheless, the solution derivatives in these points are smoothed out along the interval.

In this paper, we are concerned on numerical issues and detecting the singular points which is critical to the numerical solution of (1.1) by the piecewise collocation method so that for the detection and computation of these points we use the strategy which is essentially based on the process being offered by Guglielmi and Hairer in [14] for implicit delay differential equations.

The remainder of the paper is as follows. In the next section we will give some preliminary concepts and results regarding the continuity and smoothing properties of the solution. The numerical solvability of the (1.1) using the piecewise collocation method is investigated in Section 3. In Section 4, we obtain a general convergence theorem using an alternative approach which is conceptually due to [11] and finally Section 5 includes some test problems that illustrate the theoretical results.

2 Some basic results

In this section, we gather some well-known definitions from [11, 24] and give some basic important results, which we will use throughout the paper.

At first, we consider the following definition which is followed directly from [24] for SDDDEs with some minor corrections:

Definition 1 The SDDIE (1.1) has continuity class $p \geq 1$, if and only if the followings hold over the appropriate domain:

$i)$ All of the mixed partial derivatives of k_{ij} and θ_{ij} are continuous, for all $i + j \leq p$.
$ii)$ $g \in C^p[a, T]$, and $\phi \in C^p[\lambda, a]$.

In the following, we will show that under certain assumptions, the solution of the SDDIE (1.1) is continuous on $[a, T]$:

Theorem 1 *Let the* (1.1) *has continuity class* $p \geq 1$*, with a unique bounded solution. Furthermore the following condition holds:*

(H) $L_\theta\ \widetilde{K} < 1$*, where* L_θ *is the Lipschitz constant of* θ *with respect to the second variable,* $||y||_\infty \leq C$*,* $K = \sup\{|k(t,s)|,\ (t,s) \in [a,T] \times [\lambda, T]\}$*, and* $\widetilde{K} = KC$

then $y(t)$ *is continuous on* $[a, T]$.

Proof Let $t \in [a, T]$, we have

$$y(t^+) - y(t^-) = g(t^+) - g(t^-) + \int_{\theta(t^+, y(t^+))}^{t} k(t^+, s)y(s)ds - \int_{\theta(t^-, y(t^-))}^{t} k(t^-, s)y(s)ds.$$

Since (1.1) has continuity class $p \geq 1$, then

$$|y(t^+) - y(t^-)| = |\int_{\theta(t, y(t^+))}^{\theta(t, y(t^-))} k(t,s)y(s)ds| \leq L_\theta\ \widetilde{K}|y(t^+) - y(t^-)|.$$

According to assumption (**H**), we conclude $|y(t^+) - y(t^-)| = 0$, and this shows $y(t)$ is continuous on $[a, T]$. □

Any numerical method for SDDIEs, must address several principles concerning singular point propagation that were developed by Neves and Feldstein in [11, 24]. In what follows, we give an explanation of these issues without lengthy and detailed hypothesis.

Definition 2 (From [4, 12]) Suppose that Z_j be the zero of the equation $\theta(t, y(t)) = Z_i$, then Z_i is known as a unique ancestor of Z_j and $Z_0 = a$ is called a 0-level singular point. Consequently, Z_j is called a k-level singular point, if Z_i be a $k-1$-level singular point.

Now, let us assume that z_i is the order of continuity of Z_i, that is the greatest integer such that $y^{(z_i - 1)}$ is continuous at Z_i. Using Theorem 1, we conclude $z_i \geq 1$, for $i \geq 1$. Following [24], the discontinuities in the solution or its derivatives can occur only at zeros of the equation:

$$\begin{cases} \theta(Z_j, y(Z_j)) = Z_i, & 0 \leq i < j \leq \eta \\ Z_0 = a. \end{cases} \tag{2.1}$$

If Z_j is a zero of the equation whose multiplicity is even, then Z_j cannot be a singular point, that is $y \in C^p$ near Z_j. Noting that such a Z_j has no computational role. (For more details see [4, 24]). Since generically the multiplicity of a singular point is one and a higher multiplicity cannot be checked numerically, we here focus on the singular points with simple multiplicity. So from now on, our mean about the singular points is the simple multiplicity zeros of (2.1), with order of continuity less than or equal p.

The following auxiliary theorem which is known as *smoothing property* tells us like SDDDEs [24], the smoothing property of $y(t)$ in each singular point increases with respect to its ancestor. In fact, the solution becomes smoother and smoother as the integration proceeds:

Theorem 2 *Suppose that the* (1.1) *has a continuity class* $p \geq 1$ *and the condition* (**H**) *holds. Let* Z_j *be the least zero of* $\theta(t, y(t)) = Z_i$ *with simple multiplicity such that* $Z_j > Z_i$*, and* z_i, z_j *are the order of continuity of* Z_i *and* Z_j*, respectively, with* $0 \leq z_i < p$*, then* $z_j = z_i + 1$*.*

Proof If $z_i = 0$ then $z_j = 1$, because of Theorem 1. Now suppose that $z_i > 0$. For the sake of simplicity, we set $k(t, s)y(s) := G(t, s, y)$ and let G_i be the ith partial derivative of G with respect to the first argument, then for each $1 \leq k \leq p$:

$$y^{(k)}(t) = g^{(k)}(t) + \int_{\theta(t,y(t))}^{t} G_k(t, s, y(s))ds + \sum_{l=0}^{k-1} \frac{d^l}{dt^l}\Big[G_{K-l-1}(t, t, y(t)) - \theta^{'}(t, y(t))G_{k-l-1}(t, \theta(t, y(t)), y(\theta(t, y(t))))\Big]. \tag{2.2}$$

Note that the highest order derivative of y in the right hand of (2.2) is kth derivative, which appears in the $(k-1)$th derivative of $\theta^{'} G_{k-l-1}(t, \theta, y(\theta))$. Moving this term to the left, showing the remainder of $\frac{d^{k-1}}{dt^{k-1}}\left(\theta^{'} G_{k-l-1}(t, \theta, y(\theta))\right)$ by $\varphi(t, \theta, y(\theta))$ and rearranging the equation with respect to $y^{(k)}$, we obtain:

$$y^{(k)}(t) = \frac{1}{1 + \theta_y G(t, \theta, y(\theta))}\Bigg[g^{(k)}(t) + \int_{\theta(t,y(t))}^{t} G_k(t, s, y(s))ds - \varphi(t, \theta, y(\theta)) + \sum_{l=0}^{k-1} \frac{d^l}{dt^l} G_{k-l-1}(t, t, y(t)) - \sum_{l=0}^{k-2} \frac{d^l}{dt^l}(\theta^{'} G_{k-l-1}(t, \theta, y(\theta)))\Bigg]. \tag{2.3}$$

Since $\theta_y(t, y(t)) \leq L_\theta$, by using the condition (**H**), we conclude $|\theta_y G(t, \theta, y(\theta))| < 1$ for each $t \in [a, T]$, therefore $\theta_y G(t, \theta, y(\theta)) + 1 \neq 0$ for each $t \in [a, T]$.

Let us set $k = z_i$. Since (1.1) has a continuity class $p \geq 1$ and $y^{(z_i-1)}$ is continuous, the first four terms in (2.3) inside the bracket are continuously differentiable at $t = Z_j$. So we should prove that the last term is continuously differentiable at $t = Z_j$. It will be obtained by using a similar manner in [24] with some complicated notations which we refrain from going into details. Finally, $y^{(z_i)}$ is continuous at $t = Z_j$ and so $z_j = z_i + 1$. □

Therefore, to obtain a high order method of order p for the (1.1), we have to compute the singular points with order of continuity less than or equal p and in the case $z_0 \geq p$, it is not necessary to compute them. As a result of the theorem, we conclude that there are only finitely many computationally important singular points with order of continuity less than or equal p, which can be ordered as $a = Z_0 < Z_1 < \cdots < Z_i < \cdots < Z_\eta \leq T$, such that each Z_j is the simple multiplicity zero of the (2.1).

The next theorem which is followed from Theorem 3.2 of [24] states that if $Z > a$ be a fixed point of $\theta(t, y(t))$, then Z is the limit of an infinite sequence of singular points:

Theorem 3 *Assume that the* (1.1) *has continuity class* $p = \infty$, *and* θ *is strictly increasing for* $t \in I$, *such that it has only one fixed point* Z *on the interval* $U \subseteq I$, *then there is a sequence of singular points such that*

$$\lim_{k\to\infty} Z_k = Z.$$

3 The numerical analysis

Here, we analyze the piecewise collocation method for SDDIE (1.1) using an appropriate strategy for detecting the singular points which has taken from [14]. We will show that inserting the singular points into the set of mesh points causes the improvement of the convergence order of the proposed method.

3.1 Piecewise collocation method

Let $I_h = \{t_n : a = t_0 < t_1 < \cdots < t_N = T\}$ be a non uniform mesh on the given interval I and

$$h_n := t_{n+1} - t_n, \qquad 0 \leq n \leq N-1,$$

where the diameter of the mesh is $h = \max_n h_n$. For given integer $p \geq 1$, we define the linear space of (real) piecewise polynomials with respect to the mesh I_h as

$$S_{p-1}^{(-1)}(I_h) := \{u : \ u|_{(t_n,t_{n+1}]} \in \pi_{p-1} \ \ (0 \leq n \leq N-1)\}, \tag{3.1}$$

where π_{p-1} denotes the space of all (real) polynomials of degree not exceeding $p-1$ and $u|_{(t_n,t_{n+1}]}$ is the restriction of u on $(t_n, t_{n+1}]$. The collocation solution $u \in S_{p-1}^{(-1)}(I_h)$ can be defined by the collocation equation

$$\begin{cases} u(t) = g(t) + \displaystyle\int_{\theta(t,u(t))}^{t} k(t,s)u(s)ds, & t \in X_h, \\ u(t) = \phi(t), & \lambda \leq t < a, \end{cases} \tag{3.2}$$

where $\theta(t, u(t)) \leq t$, for all $t \in X_h$ and

$$X_h := \left\{t_{n,i} = t_n + c_i h_n : \quad 0 \leq c_1 < \cdots < c_p \leq 1, \quad (0 \leq n \leq N-1)\right\},$$

is the set of collocation points including the collocation parameters $\{c_i\}$. For $p \geq 2$, by choosing $c_1 = 0$, $c_p = 1$, it follows that the collocation solution $u(t)$ is continuous on I.

The collocation (3.2) has the singular points similar to the original equation, namely $a = Z_{h,0} < Z_{h,1} < \cdots < Z_{h,i} < \cdots < Z_{h,\eta} \leq T$, whose levels are $\leq p$ and can be obtained from the following equation:

$$\begin{cases} \theta(t, u(t)) = Z_{h,i}, & 0 \leq i < j \leq \eta, \\ Z_{h,0} = a. \end{cases} \tag{3.3}$$

A convenient computational form of the collocation equation is obtained when we employ the local Lagrange basis functions and setting

$$\begin{cases} L_j(v) = \prod_{\substack{k=1 \\ k \neq j}}^{p} \dfrac{v - c_k}{c_j - c_k}, & v \in [0, 1] \\ \\ U_{n,j} = u(t_n + c_j h_n), \end{cases} \tag{3.4}$$

the restriction of the collocation solution to the subinterval $(t_n, t_{n+1}]$, i.e. $u_n(t)$, can be expressed as:

$$u_n(t) = u(t_n + vh_n) = \sum_{j=1}^{p} L_j(v) U_{n,j}, \qquad v \in (0, 1]. \tag{3.5}$$

Since $\theta(t_{n,i}, U_{n,i}) \leq t_{n,i}$, there exist an index $\kappa \leq n$, such that:

$$t_\kappa \leq \theta(t_{n,i}, U_{n,i}) < t_{\kappa+1}, \qquad (\kappa = \kappa(n, i)). \tag{3.6}$$

Now we consider the following two cases:

Case 1 $\kappa = n$, the collocation equation may be written as

$$U_{n,i} = g(t_{n,i}) + \int_{\theta(t_{n,i}, U_{n,i})}^{t_{n,i}} k(t_{n,i}, s) u_n(s) ds.$$

Using some manipulations we conclude:

$$U_{n,i} = g_{n,i} + h_n \sum_{j=1}^{p} \left(\int_{\frac{\theta(t_{n,i}, U_{n,i}) - t_n}{h_n}}^{c_i} k(t_{n,i}, t_n + sh_n) L_j(s) ds \right) U_{n,j},$$

or equivalently, in the compact form:

$$U_n = g_n + h_n \bar{\Gamma}_n(U_n) U_n, \tag{3.7}$$

where

$$\left[\bar{\Gamma}_n(U_n)\right]_{ij} := \int_{\frac{\theta(t_{n,i}, U_{n,i}) - t_n}{h_n}}^{c_i} k(t_{n,i}, t_n + sh_n) L_j(s) ds,$$

and

$$U_n = (U_{n,1}, ..., U_{n,p})^T, \qquad g_n = (g_{n,1}, ..., g_{n,p})^T, \qquad g_{n,i} = g(t_{n,i}).$$

Case 2 $\kappa < n$, in this case we have

$$U_{n,i} = g_{n,i} + \int_{\theta(t_{n,i},U_{n,i})}^{t_{\kappa+1}} k(t_{n,i}, s)u_\kappa(s)ds + \sum_{l=\kappa+1}^{n-1} \int_{t_l}^{t_{l+1}} k(t_{n,i}, s)u_l(s)ds + \int_{t_n}^{t_{n,i}} k(t_{n,i}, s)u_n(s)ds,$$

and so

$$\begin{aligned} U_{n,i} &= g_{n,i} + h_\kappa \sum_{j=1}^{p} \left(\int_{\frac{\theta(t_{n,i},U_{n,i})-t_\kappa}{h_\kappa}}^{1} k(t_{n,i}, t_\kappa + sh_\kappa)L_j(s)ds \right) U_{\kappa,j} \\ &+ \sum_{l=\kappa+1}^{n-1} h_l \sum_{j=1}^{p} \left(\int_0^1 k(t_{n,i}, t_l + sh_l)L_j(s)ds \right) U_{l,j} \\ &+ h_n \sum_{j=1}^{p} \left(\int_0^{c_i} k(t_{n,i}, t_n + sh_n)L_j(s)ds \right) U_{n,j}. \end{aligned}$$

Therefore:

$$U_n = g_n + h_\kappa \Gamma_n(U_n)U_\kappa + \sum_{l=\kappa+1}^{n-1} h_l \mathbf{B}_n^{(l)} U_l + h_n \mathbf{B}_n U_n, \qquad n = 0, 1, 2, ..., N-1, \quad (3.8)$$

in which

$$\begin{aligned} [\Gamma_n(U_n)]_{ij} &:= \int_{\frac{\theta(t_{n,i},U_{n,i})-t_\kappa}{h_\kappa}}^{1} k(t_{n,i}, t_\kappa + sh_\kappa)L_j(s)ds, \\ [\mathbf{B}_n]_{ij} &:= \int_0^{c_i} k(t_{n,i}, t_n + sh_n)L_j(s)ds, \\ \left[\mathbf{B}_n^{(l)}\right]_{ij} &:= \int_0^1 k(t_{n,i}, t_l + sh_l)L_j(s)ds, \qquad (i, j = 1, 2, ..., p). \end{aligned}$$

Note that, in both cases we end up with an algebraic nonlinear system of equations with $U_n = (U_{n,1}, ..., U_{n,p})^T$ unknowns, whose solution with (3.5) give the collocation solution $u(t)$. Generally, the integrals occurring in the collocation equation can not be obtained in analytical form, and a suitable quadrature is needed for their approximations.

The discretized form of (3.2) can be considered as

$$\hat{u}(t) = g(t) + (\hat{w}_\theta \hat{u})(t), \qquad t \in X_h,$$

where $\hat{w}_\theta$ is a discretization of w_θ in the (1.1). Following [6], we need to make sure that the quadrature formulas are chosen such that the order of the resulting quadrature errors will match the order of convergence of the exact collocation solution. This can be achieved if we choose interpolatory p-point quadrature formulas whose abscissas are given by the p collocation parameters $\{c_i\}$. In particular, the general quadrature rule for an arbitrary function $f(s)$, for $s \in [0, 1]$ may be considered as

$$\int_0^1 f(s)ds \approx \sum_{r=1}^{p} b_r f(c_r),$$

where $b_r = \int_0^1 L_r(s)ds$. Therefore

$$\int_0^1 k(t_{n,i}, t_l + sh_l)L_j(s)ds \approx \sum_{r=1}^{p} b_r k(t_{n,i}, t_{l,r})L_j(c_r) = b_j k(t_{n,i}, t_{l,j}) := \left[\hat{B}_n^{(l)}\right]_{ij},$$

and

$$\begin{aligned}\int_0^{c_i} k(t_{n,i}, t_n + sh_n)L_j(s)ds &= c_i \int_0^1 k(t_{n,i}, t_n + c_i sh_n)L_j(c_i s)ds \\ &\approx c_i \sum_{r=1}^{p} b_r k(t_{n,i}, t_n + c_i c_r h_n)L_j(c_i c_r) := \left[\hat{B}_n\right]_{ij}.\end{aligned}$$

Consequently, the other integrals can be approximated as follows:

$$\begin{aligned}\left[\hat{\Gamma}_n(U_n)\right]_{ij} &:= \frac{t_{\kappa+1} - \theta(t_{n,i}, U_{n,i})}{h_\kappa} \sum_{r=1}^{p} b_r k(t_{n,i}, (1 - c_r)\theta(t_{n,i}, U_{n,i}) + c_r t_{\kappa+1}) \\ &\times\ L_j\left((1 - c_r)\left(\frac{\theta(t_{n,i}, U_{n,i}) - t_\kappa}{h_\kappa}\right) + c_r\right), \\ \left[\hat{\bar{\Gamma}}_n(U_n)\right]_{ij} &:= \frac{t_{n,i} - \theta(t_{n,i}, U_{n,i})}{h_n} \sum_{r=1}^{p} b_r k(t_{n,i}, c_r t_{n,i} + (1 - c_r)\theta(t_{n,i}, U_{n,i})) \\ &\times\ L_j\left(c_i c_r + (1 - c_r)\left(\frac{\theta(t_{n,i}, U_{n,i}) - t_n}{h_n}\right)\right),\end{aligned}$$

where $\hat{B}_n^{(l)}$, $\hat{B}_n$, $\hat{\Gamma}_n(U_n)$, and $\hat{\bar{\Gamma}}_n(U_n)$ are the discretization form of $\mathbf{B}_n^{(l)}$, $\mathbf{B}_n$, $\Gamma_n(U_n)$, and $\bar{\Gamma}_n(U_n)$, respectively.

Under these assumptions, the fully discretized collocation equations in both cases are respectively as follows:

$$\hat{U}_n = g_n + h_n \hat{\bar{\Gamma}}_n(\hat{U}_n)\hat{U}_n,$$

$$\hat{U}_n = g_n + h_\kappa \hat{\Gamma}_n(\hat{U}_n)\hat{U}_\kappa + \sum_{l=\kappa+1}^{n-1} h_l \hat{B}_n^{(l)}\hat{U}_l + h_n \hat{B}_n \hat{U}_n,$$

where $\hat{U}_n = (\hat{U}_{n,1}, ..., \hat{U}_{n,p})^T$ and $\hat{U}_{n,i} = \hat{u}_n(t_{n,i})$, for $n = 0, 1, 2, ..., N - 1$.

3.2 Numerical algorithms

It is known, the efficiency of a numerical method for solving the state dependent delay integral equations can be improved significantly if the singular points are chosen carefully, like SDDDEs. In the SDDIEs under consideration, the singular points can not be computed in advance, therefore we should obtain the collocation solutions and the singular points simultaneously.

There are some different strategies in the literature for detection and computation of the singular points mainly associated to SDDDEs. (see e.g. [11, 14, 18]). Here, we use an efficient procedure for detection and computation of the singular points for implicit delay differential equations which is mainly based on the paper of Guglielmi and Hairer [14].

In our numerical algorithms, all singular points with level less than or equal p are computed to get a method with order of convergence p. We start with $t_0 := a$ and $t_1 := t_0 + h$, where h is an initial step size. Suppose that the problem is solved successfully until t_n, and the singular point $Z_{h,i}$ is computed in the previous steps. Our aim is computing the $Z_{h,j}$. Let us define $f(t) = \theta(t, u_{n-1}(t)) - Z_{h,i}$, where $u_{n-1}(t)$ is the collocation solution of the preceding step. We consider two cases:

Case 1 If $f(t)$ does not change sign on $[t_n, t_n + h]$, then we take $t_{n+1} := t_n + h$ and solve the collocation (3.7)–(3.8) for the unknowns $U_{n,1}, ..., U_{n,p}$ using a Newton type method, while it is necessary to choose suitable initial values for unknowns with desired accuracy (we take $U_{n,i}^{[0]} := u_{n-1}(t_{n,i})$). Noting that in the initial point t_0 ($z_0 = 0$), the choice of $u_{-1}(t)$ as an approximation of $u_0(t)$ may be unsuitable and so the considered iterative method may be divergent. To overcome this difficulty we may use $u_0(t_0^+)$ as an approximation of $u_0(t_{0,i})$ such that:

$$u_0(t_0^+) = g(t_0) + \int_{\theta(t_0, u_0(t_0^+))}^{t_0} k(t_0, s)\phi(s)ds.$$

There are unknown indices κ in the collocation equations which should be determined, however according to (3.6) each κ depends on the stage value $U_{n,i}$. Therefore, in each step of the iterative method, the value of each $\kappa_{n,i}$ obtains by using the value of $U_{n,i}$ in the previous step.

We summarize our strategy in the following algorithm:

Algorithm 1

for $l = 1, 2, ...$ **until** convergence.
begin
- Compute $\kappa_{n,i}^{[l]}$, by replacing $U_{n,i}$ with $U_{n,i}^{[l-1]}$ in relation (3.6), for each $i = 1, ..., p$;
- Take $\kappa_{n,i} := \kappa_{n,i}^{[l]}$ in the collocation equations and apply one step of the iterative method to compute $U_{n,1}^{[l]}, ..., U_{n,p}^{[l]}$;

end.

Case 2 If $f(t)$ changes sign on $[t_n, t_n+h]$, then we consider t_{n+1} (the next grid point) as an unknown value in the collocation (3.7)–(3.8) and determine it, in such a way that t_{n+1} exactly equals to Z_{hj}. For this purpose, we denote the collocation (3.7)–(3.8) by $F_i(U_{n,1}, ..., U_{n,p}, t_{n+1}, \kappa_{n,1}, ..., \kappa_{n,p}) = 0$, where $U_{n,1}, ..., U_{n,p}$, and t_{n+1} are unknowns and each $\kappa_{n,i}$ depends on the stage values $U_{n,i}$, which is determined by the (3.6). Now, we consider the following system of equations for the unknowns $U_{n,1}, ..., U_{n,p}$ and t_{n+1}

$$F_i(U_{n,1}, ..., U_{n,p}, t_{n+1}) = 0, \qquad i = 1, 2, ..., p, \tag{3.9}$$

$$\theta(t_{n+1}, u_n(t_{n+1})) = Z_{h,i}, \tag{3.10}$$

where $u_n(t)$ is the collocation solution of the current step that depends on the $U_{n,1}, ..., U_{n,p}$. The above system may be solved using a Newton type method, that exploits the structure of the system. With the aim of not destroying this structure, we solve the above system with a *splitting idea*. (See e.g. [13, 14]). While, it is necessary to choose suitable initial values for unknowns with desired accuracy ($U_{n,i}^{[0]} := u_{n-1}(t_{n,i})$) and use the zero of $\theta(t, u_{n-1}(t)) = Z_{h,i}$ on $[t_n, t_n + h]$ as initial value of t_{n+1}.

The following algorithm summarizes our strategy in this case:

Algorithm 2

for $l = 1, 2, ...$ **until** convergence.
begin
- Compute $\kappa_{n,i}^{[l]}$, by replacing $U_{n,i}$ with $U_{n,i}^{[l-1]}$ in relation (3.6), for each $i = 1, ..., p$;
- Take $\kappa_{n,i} := \kappa_{n,i}^{[l]}$ and $t_{n+1} := t_{n+1}^{[l-1]}$ in (3.9) and apply one step of the iterative method which yields collocation approximation $u_n^{[l]}(t_n + vh_n)$ for $v \in [0, 1]$;
- Replacing $u_n(t)$ with $u_n^{[l]}(t)$ in the (3.10) and compute $t_{n+1}^{[l]}$ by using one step of the iterative method on this equation;

end.

Remark 1 If there are more than one singular point in the interval $[t_n, t_{n+1}]$, such that the algorithm can not immediately find the leftmost one, the error control method [13, 25] should be used to automatically rejects the step size and accordingly reduces it.

4 Convergence analysis

The object of the present section is to describe the convergence behavior of the proposed numerical method. We will show that under appropriate conditions, when the singular points are approximated by the presented process in Section 3.2, the rate of convergence u to the exact solution y depends on the continuity conditions of the (1.1), i.e. if this equation has continuity class p, then the collocation solution has order of convergence p.

In order to obtain the convergence order, it is necessary to locate primary singular points as well. To do so, we divide our discussion into two folds: vanishing and non-vanishing delays.

In the first one, using Theorem 3, it is seen how the singular points are located near the zero point of the delay function $\theta(t, y(t)) - t$. In this case, which is called *vanishing delay*, there exist a point $Z > a$ such that $\theta(Z, y(Z)) = Z$. As in equation (1.1), it is assumed that $\theta(.,.)$ is continuous and strictly increasing function, for any k level singular point Z_i, where $\theta(Z_i, y(Z_i)) < Z_i$, then there exist $(k+1)$ level singular point Z_j, such that $\theta(Z_j, y(Z_j)) < Z_j$. From Theorem 3, we have also $Z_i < Z_j < Z$, so there are infinitely many singular points on the left hand side of Z. To achieve a numerical method of pth order, we have to locate the points that the mth derivatives ($m \le p$) have really jump. According to Theorems 2 and 3, we deduce that $y \in C^{(m)};\ m > p$, in a neighborhood of Z. Due to the fact that our computational concern is jump points satisfied in the relation $y^{(m)}(t), m \le p$, without loss of generality, we may ignore such neighborhoods of fixed point Z. Therefore, the different continuity orders of these points yield, there exist many primary finitely singular points which are isolated. In the *non-vanishing delay*, which is assumed that $\theta(t, y(t)) < t$, we have not any clustering fixed point. Therefore according to Theorem 2, there exist many primary finitely singular points which are isolated as well.

For the given singular points Z_i and Z_j, there exist ξ_i and ξ_j such that

$$a \le Z_i - \xi_i < Z_i + \xi_i < Z_j - \xi_j < Z_j + \xi_j \le T,$$

where Z_i is an ancestor of Z_j.

Let $\hat{y}, \hat{\hat{y}}$ be two C^p extensions of y, such that the first is defined from $[Z_i, Z_i + \xi_i]$ back to $[Z_i - \xi_i, Z_i + \xi_i]$, and the second is defined from $[Z_i - \xi_i, Z_i]$ forward to $[Z_i - \xi_i, Z_i + \xi_i]$. The following Lemma from [11] is useful to obtaining the collocation error at the near singular points.

Lemma 1 *If the* (1.1) *has continuity class $p \ge 1$, then for all sufficiently small $h > 0$,*

$$\|\hat{y} - y\|_{\infty}^{[Z_i, Z_i + h]} = O(h^{z_i}),$$

$$\|\hat{\hat{y}} - y\|_{\infty}^{[Z_i - h, Z_i]} = O(h^{z_i}),$$

where z_i is the order of continuity of Z_i.

Now, let Z_{hi} denotes an approximate value of Z_i, which is generated by the introduced approach and r_i is the rate of error, i.e. $|Z_{hi} - Z_i| = O(h^{r_i})$. The following theorem is the main result of this section:

Theorem 4 *Let the SDDIE* (1.1) *has continuity class $p \ge 1$, the condition* (**H**) *holds and the equation has only one singular point $Z_i \in [t_m, t_m + h) \subseteq [t_0, t_M],\ M \le N$.*

If $Z_{hi} \in [t_m, t_m + h)$, *then*

$$\|y - u\|_{\infty}^{[t_0, t_M]} = O(h^{\min\{p, r_i z_i\}}),$$

where u is the collocation approximation of y.

Proof Let us assume that $Z_{hi} < Z_i$ and Z_{hi} be a nodal point i.e. $t_{m+1} = Z_{hi}$ and $t_{m+2} = t_m + h$, also let Θ be the set of all integer numbers from $\hat{k}$ to M where $\hat{k} = \max\{\kappa(n, i) : n = 0, 1, ..., M-1, i = 1, 2, ..., p\}$ where each $\kappa(n, i)$ satisfies (3.6). We consider the following two cases:

Case 1 $l \in \Theta - \{m+1\}$, since there is not any singular points on $[t_l, t_{l+1}]$, we conclude $y \in C^p[t_l, t_{l+1}]$. By using the Peano's Theorem we have

$$y(t_l + sh_l) = \sum_{j=1}^{p} L_j(s) Y_{l,j} + h_l^p R_{p,l}(y, s), \quad s \in [0, 1], \tag{4.1}$$

where $Y_{lj} = y(t_{lj})$ and $R_{p,l}(y, s) = \int_0^1 k_p(s, z) y^{(p)}(t_l + zh_l) dz$, with

$$k_p(s, z) = \frac{1}{(p-1)!} \left\{ (s-z)_+^{p-1} - \sum_{j=1}^{p} L_j(s)(c_j - z)_+^{p-1} \right\}, \qquad z \in [0, 1].$$

It follows from (3.5) and (4.1), the collocation error $e := y - u$ possesses the local representation

$$e(t_l + sh_l) = \sum_{j=1}^{p} L_j(s) \varepsilon_{lj} + h_l^p R_{p,l}(y, s), \qquad s \in (0, 1] \tag{4.2}$$

with $\varepsilon_{lj} := Y_{lj} - U_{lj}$. Therefore

$$|e(t_l + sh_l)| \leq \Lambda_p \|\varepsilon_l\|_1 + h_l^p K_p \|y^{(p)}\|_{\infty} \leq \Lambda_p \|\varepsilon_l\|_1 + O\left(h_l^p\right), \tag{4.3}$$

where $\Lambda_p = \max_{(j)} \|L_j\|_{\infty}$ and $K_p = \max_{s \in [0,1]} \int_0^1 |k_p(s, z)| dz$. □

Case 2 $l = m + 1$, and $\hat{y}$ is a C^p extension of y from $[Z_i, t_{m+2}]$ back to $[Z_{hi}, t_{m+2}]$, consequently using the Peano's Theorem and Lemma 1, we have

$$\begin{aligned} |e(t_{m+1} + sh_{m+1})| &\leq |y(t_{m+1} + sh_{m+1}) - \hat{y}(t_{m+1} + sh_{m+1})| \\ &\quad + |\hat{y}(t_{m+1} + sh_{m+1}) - u(t_{m+1} + sh_{m+1})| \\ &\leq \|y - \hat{y}\|_{\infty}^{[Z_{hi}, Z_i]} + \sum_{j=1}^{p} |L_j(s)||\hat{\varepsilon}_{m+1,j}| + h_{m+1}^p |R_{p,m+1}(\hat{y}, s)| \\ &= O(h^{r_i z_i}) + \sum_{j=1}^{p} |L_j(s)||\hat{\varepsilon}_{m+1,j}| + O(h_{m+1}^p), \end{aligned} \tag{4.4}$$

where $\hat{\varepsilon}_{m+1,j} = \hat{y}(t_{m+1,j}) - u(t_{m+1,j})$. Representing $\hat{\varepsilon}_{m+1,j} - \varepsilon_{m+1,j} = \hat{y}(t_{m+1,j}) - y(t_{m+1,j})$, we have

$$|\hat{\varepsilon}_{m+1,j}| \le |\varepsilon_{m+1,j}| + |\hat{y}(t_{m+1,j}) - y(t_{m+1,j})| \le |\varepsilon_{m+1,j}| + O(h^{r_i z_i}),$$

thus

$$|\hat{\varepsilon}_{m+1,j}| \le |\varepsilon_{m+1,j}| + O(h^{r_i z_i}). \tag{4.5}$$

Inserting (4.5) into (4.4) yields

$$|e(t_{m+1} + sh_{m+1})| \le \Lambda_p \|\varepsilon_{m+1}\|_1 + O(h_{m+1}^p) + O(h^{r_i z_i}). \tag{4.6}$$

From (1.1) and (3.2) we have

$$\begin{aligned}
\varepsilon_{n,i} = & \int_{\theta(t_{n,i},Y_{n,i})}^{\theta(t_{n,i},U_{n,i})} k(t_{n,i}, s)y(s)ds + \int_{\theta(t_{n,i},U_{n,i})}^{t_{\kappa+1}} k(t_{n,i}, s)e(s)ds \\
& + \sum_{l=\kappa+1}^{n-1} \int_{t_l}^{t_{l+1}} k(t_{n,i}, s)e(s)ds + \int_{t_n}^{t_{n,i}} k(t_{n,i}, s)e(s)ds. \quad (n = 0, 1, ..., M-1)
\end{aligned}$$

Using some manipulations, the above relation with the condition (**H**) yield

$$|\varepsilon_{ni}| \le \frac{K}{1 - \widetilde{K} L_\theta} \left[\sum_{l=\kappa}^{n-1} h_l \int_0^1 |e(t_l + sh_l)|ds + h_n \int_0^{c_i} |e(t_n + sh_n)|ds \right].$$

Now, by using (4.3) and (4.6) we have

$$\begin{aligned}
|\varepsilon_{ni}| \le & \ \zeta \sum_{l=\kappa}^{n-1} h_l \int_0^1 [\Lambda_p \|\varepsilon_l\|_1 + O(h_l^p)]ds + h_n \int_0^{c_i} \left[\Lambda_p \|\varepsilon_n\|_1 + O(h_n^p)\right] ds \\
& + f_{n,m} \ O(h^{rz+1}),
\end{aligned}$$

where $\zeta = \frac{K}{1 - \widetilde{K} L_\theta}$, $f_{n,m} = 1$ for $n \ge m$ and otherwise $f_{n,m} = 0$. This gives

$$(1 - h_n p \Lambda_p \zeta)\|\varepsilon_n\|_1 \le p\ \zeta \left[\Lambda_p \sum_{l=\kappa}^{n-1} h_l \|\varepsilon_l\|_1 + \sum_{l=\kappa}^{n} O(h_l^{p+1}) \right] + f_{n,m}\ O(h^{r_i z_i + 1}).$$

According to condition (**H**) and choosing the step size h so small, we obtain $h_n p \Lambda_p \zeta < 1$, therefore

$$\|\varepsilon_n\|_1 \le C_1 \sum_{l=\kappa}^{n-1} h_l \|\varepsilon_l\|_1 + C_2 O(h^{p+1}) + f_{n,m} O(h^{r_i z_i + 1}), \tag{4.7}$$

From Gronwall's inequality [4], we have

$$\|\varepsilon_n\|_1 \le O(h^{p+1}) + f_{n,m}\ O(h^{r_i z_i + 1}). \tag{4.8}$$

Inserting (4.8) into (4.3) or (4.6) yield

$$|e(t_n + sh_n)| \le O(h^p) + f_{n,m}\ O(h^{r_i z_i}). \qquad (n = 0, 1, .., M-1)$$

Finally, the error estimate of the theorem will be obtained □

Theorem 5 *Assume the hypothesis of Theorem 4 hold. Furthermore let* $Z_j \in [t_m, t_m + h]$ *and* $z_j r_j \geq p$*, then for sufficiently small* h

$$|Z_j - Z_{hj}| = O(h^{\min\{p, r_i\}}).$$

Proof Let us set $f(t) = \theta(t, y(t)) - Z_i$ and $f_h(t) = \theta(t, u(t)) - Z_{hi}$. By using mean value theorem for each $t \in [t_m, t_m + h]$, we can write

$$f(t) = \big(t - Z_j\big)\, f'(\xi(t)), \qquad \text{for some } \xi(t) \in (t_m, t_m + h), \tag{4.9}$$

hence $f(t) = O(h)$. Due to the Lipschitz continuity of θ with respect to its second variable, Theorem 4 and $r_j z_j \geq p$, we have

$$|f(t) - f_h(t)| \leq L_\theta |y(t) - u(t)| + |Z_i - Z_{hi}| = O(h^{p, z_j r_j}) + O(h^{r_i}) = O(h^{\min\{p, r_i\}}). \tag{4.10}$$

Since h is sufficiently small, using $f(t_m) = O(h)$ and (4.10) we have

$$|f(t_m) - f_h(t_m)| \leq |f(t_m)|.$$

This implies that f and f_h have the same sign at t_m and similarly at $t_m + h$. Due to changing sign of f on $[t_m, t_m + h]$, f_h also changes sign on $[t_m, t_m + h]$. Since $f_h(Z_{hj}) = 0$ and f_h is continuous on $[t_m, t_m + h]$, then $Z_{hj} \in [t_m, t_m + h)$. Now by using Theorem 2, we conclude that $z_j \geq 1$, and so $f'(t)$ is continuous on $t = Z_j$. Considering the continuity of $f'(t)$ and the fact that $f'(Z_j) \neq 0$ for sufficiently small h, there exists an open interval $V \subset I$ such that $Z_{hj}, Z_j \in V$ and $|f'(t)| > 0$ for all $t \in \overline{V}$. Therefore, there exists a minimum M for $|f'(t)|$ on $\overline{V}$, such that

$$|f'(t)| \geq M > 0, \qquad \text{for all } t \in \overline{V}. \tag{4.11}$$

From (4.10) we get

$$|f(Z_{hj})| = O(h^{\min\{p, r_i\}}). \tag{4.12}$$

It follows from (4.9), (4.12) and inequality (4.11) that

$$|Z_{hj} - Z_j| = \frac{|f(Z_{hj})|}{|f'(\xi)|} = O(h^{\min\{p, r_i\}}).$$

□

Corollary 1 *Let the* (1.1) *has continuity class* $p \geq 1$*, and there exist a finite number of singular points with order of continuity lees than* p *in* $[a, T]$*, which are ordered as,* $a = Z_0 < Z_1 < \cdots < Z_\eta \leq T$*. Repeated application of Theorem 4, gives*

$$\|y - u\|_\infty^{[a,T]} = O(h^Q), \tag{4.13}$$

where

$$Q = \min_{0 \leq j \leq \eta} \{p, r_j z_j\}.$$

To complete this section, we give the following theorem which shows that, when the singular points are included in the mesh points, the high order of convergence is maintained as well.

Theorem 6 *Assume that* (1.1) *has continuity class $p \geq 1$, and there exist a finite number of singular points with order of continuity lees than p in $[a, T]$, which are ordered as, $a = Z_0 < Z_1 < \cdots < Z_\eta \leq T$. Furthermore, let $z_0 \geq 1$, condition* (**H**) *holds and the singular points are approximated by the method described in Section 3.2, then*

$$\|y - u\|_\infty^{[a,T]} = O(h^p).$$

Proof The proof is followed from Theorem 5.4 in [11]. □

5 Numerical results

In this section, we present results of some numerical experiments to illustrate the effectiveness of the proposed method. We consider the collocation parameters as $c_i = (i-1)/(p-1)$, for $i = 1, 2, ..., p$ in Examples 1, 2, 4 and Gauss Lobatto points in Example 3. The accuracy of the proposed method is measured by computing the maximum errors at overall interval I. This shows that after computing the singular points, the proposed method maintains the convergence order p. We also report the observed order of convergence of the method. All computations are performed by Maple® 12 with tolerance 10^{-10}. It should be noted that, in all the examples we have firstly considered the equal step sizes and after detecting the singular points we transit to the variable case.

Example 1 Consider the following SDDIE

$$\begin{cases} y(t) = \sqrt{2} + \int_{\ln(y(t))-1}^{t} y(s)ds, & 0 \leq t \leq 1 \\ y(t) = 0. & t \leq 0 \end{cases}$$

with the exact solution

$$y(t) = \begin{cases} \sqrt{2}\exp(t), & 0 \leq t \leq Z_1 \\ \exp(\frac{e}{e+\sqrt{2}}(t-1+\ln(2)/2)+1), & Z_1 \leq t \leq 1 \end{cases}$$

Table 1 The errors $|Z_1 - Z_{h1}|$ and $\|y-u\|_\infty^{[0,1]}$ in Example 1

h	$\lvert Z_1 - Z_{h1}\rvert$		$\|y-u\|_\infty^{[0,1]}$	
	$p=2$	$p=4$	$p=2$	$p=4$
h_0	2.91E−03	3.15E−07	2.05E−02	4.74E−06
$h_0/2$	8.17E−04	2.36E−08	6.70E−03	3.70E−07
$h_0/4$	2.05E−04	1.48E−09	1.75E−03	2.39E−08
$h_0/8$	5.28E−05	9.38E−11	4.47E−04	1.51E−09
$h_0/16$	1.32E−05	5.90E−12	1.16E−04	9.73E−11
$h_0/32$	3.31E−06	3.72E−13	2.94E−05	6.17E−12

Table 2 Convergence orders of $|Z_1 - Z_{h1}|$ and $\|y-u\|_\infty^{[0,1]}$ in Example 1

h	$\lvert Z_1 - Z_{h1}\rvert$		$\|y-u\|_\infty^{[0,1]}$	
	$p=2$	$p=4$	$p=2$	$p=4$
h_0	1.83	3.74	1.61	3.68
$h_0/2$	1.99	4.00	1.94	3.95
$h_0/4$	1.96	3.97	1.97	3.98
$h_0/8$	2.00	3.99	1.95	3.96
$h_0/16$	2.00	3.99	1.98	3.98
$h_0/32$	2.00	3.99	1.99	3.99

which is obtained by using the method of steps which introduced by Bellman [5], where $Z_1 = 1 - \frac{1}{2}\ln(2) \approx 0.6534264$. The maximum error $\|y-u\|_\infty^{[0,1]}$ and absolute error $|Z_1 - Z_{h1}|$ are tabulated in Table 1, for different values of p and h with $h_0 = 0.25$.

Here, $Z_0 = t_0 = 0$, $z_0 = 0$, $r_0 = \infty$, $z_1 = 1$, $r_1 = \min\{p, r_0\} = p$, so $|Z_1 - Z_{h1}| = \min\{p, r_0\} = p$, which can be observed from Table 2. Also from (4.13), we conclude that $Q = \min\{p, z_0r_0, z_1r_1\} = p$, where z_0r_0 is determined from $\|y-\hat{y}\|_\infty^{[Z_{h0},Z_0]} = O(h^{r_0z_0}) = O(h^\infty)$, because of $Z_0 = Z_{h0}$. Reported numerical results in Table 2 confirm the results of Theorems 4 and 5.

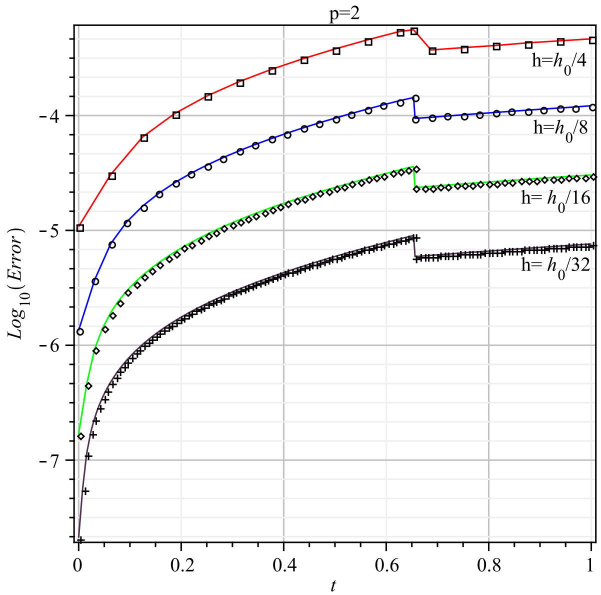

Fig. 1 Point-wise absolute errors for $p = 2$, and different values of h in Example 1

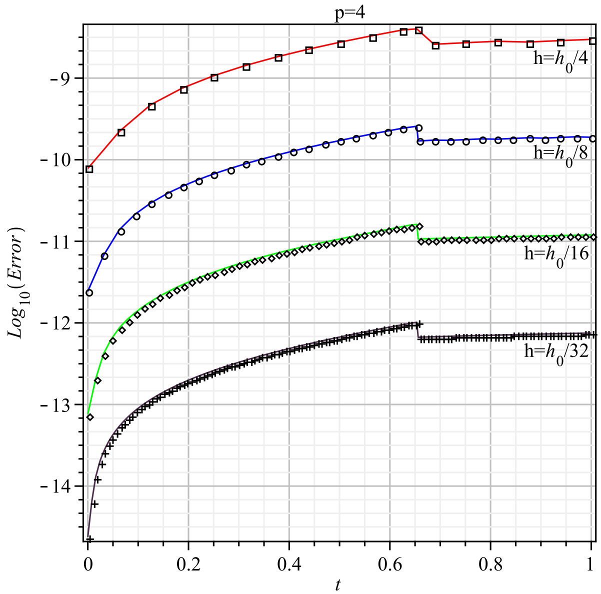

Fig. 2 Point-wise absolute errors for $p = 4$, and different values of h in Example 1

Figures 1 and 2 show the point-wise absolute errors $|y(t) - u(t)|$ in the collocation points $t_{n,i}$, for different values of h with $p = 2, \ 4$. They indicate that the point-wise absolute errors decay exponentially as h decreases. We observe that there is a jump in the error behaviors at the singular point $Z_1 \approx 0.6534264$.

Example 2 Consider the following SDDIE with two singular points

$$\begin{cases} y(t) = 1 + \displaystyle\int_{y(t)-\sqrt{2}}^{t} \frac{y(s)}{s+1} ds, & 0 \leq t \leq 2 \\ y(t) = 0, & t < 0 \end{cases} \tag{5.1}$$

Table 3 The errors $|Z_1 - Z_{h1}|, \ |Z_2 - Z_{h2}|$ and $\|y - u\|_\infty^{[0, Z_{h2}]}$ in Example 2

h	$\lvert Z_1 - Z_{h1}\rvert$		$\lvert Z_2 - Z_{h2}\rvert$		$\\|y - u\\|_\infty^{[0, Z_{h2}]}$	
	$p = 2$	$p = 4$	$p = 2$	$p = 4$	$p = 2$	$p = 4$
h_0	9.71E−17	9.72E−17	1.18E−03	3.57E−07	1.03E−03	4.31E−07
$h_0/2$	9.72E−17	9.72E−17	3.32E−04	2.51E−08	2.76E−04	2.96E−08
$h_0/4$	9.72E−17	9.72E−17	8.79E−05	1.82E−09	7.96E−05	2.31E−09
$h_0/8$	9.72E−17	9.72E−17	2.25E−05	1.17E−10	2.03E−05	1.48E−10
$h_0/16$	9.72E−17	9.72E−17	5.71E−06	7.57E−12	5.27E−06	9.81E−12
$h_0/32$	9.72E−17	9.72E−17	1.45E−06	4.85E−13	1.32E−06	6.15E−13

Table 4 Convergence orders of $\|y-u\|_{\infty}^{[0,Z_{h2}]}$ and $|Z_2 - Z_{h2}|$ in Example 2

h	$\|Z_2 - Z_{h2}\|$		$\|y-u\|_{\infty}^{[0,Z_{h2}]}$	
	$p=2$	$p=4$	$p=2$	$p=4$
h_0	1.83	3.83	1.90	3.86
$h_0/2$	1.92	3.79	1.79	3.68
$h_0/4$	1.96	3.96	1.97	3.96
$h_0/8$	1.98	3.95	1.95	3.92
$h_0/16$	1.98	3.97	2.00	4.00
$h_0/32$	1.99	3.98	1.98	3.98

with the exact solution:

$$y(t) = \begin{cases} t+1, & 0 \le t \le \sqrt{2}-1 \\ \sqrt[4]{2}\sqrt{t+1}. & \sqrt{2}-1 \le t \le \frac{(2\sqrt{2}-1)^2}{\sqrt{2}} - 1 \end{cases} \tag{5.2}$$

Here, the singular points are $Z_1 = \sqrt{2}-1 \approx 0.4142136$ and $Z_2 = \frac{(2\sqrt{2}-1)^2}{\sqrt{2}} - 1 \approx 1.3639610$. This problem is solved with the **second** and **fourth** order of the pro-

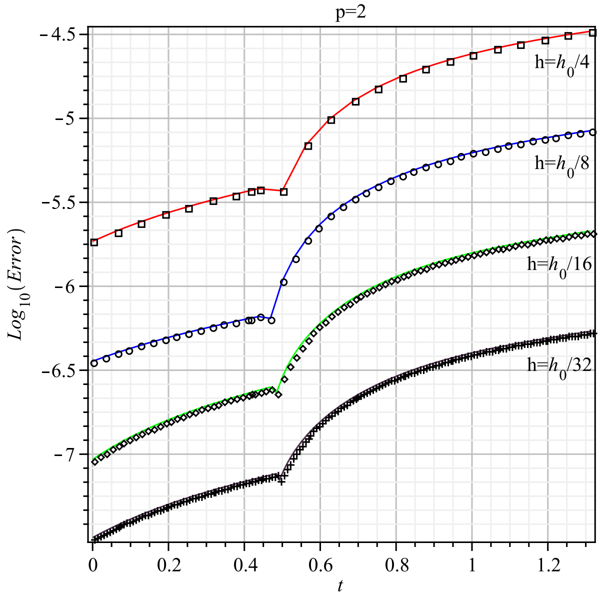

Fig. 3 Point-wise absolute errors for $p=2$, and different values of h in Example 2

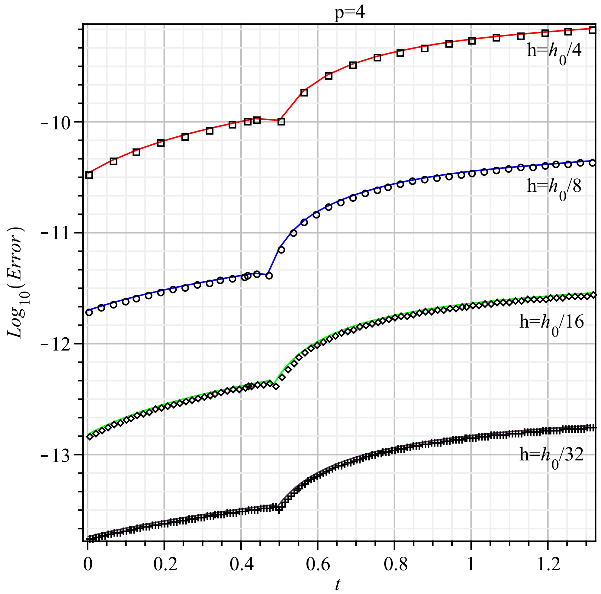

Fig. 4 Point-wise absolute errors for $p = 4$, and different values of h in Example 2

posed method. Our numerical results show that the orders of convergence 2 and 4 are achieved, respectively. In Tables 3 and 4, we have presented the overall maximum errors and the convergence orders for different values of h with $h_0 = 0.25$ and $p = 2,\ 4$. Also, Figures 3 and 4 represent the error behaviors versus the grid points.

Example 3

$$\begin{cases} y(t) = \exp(-t) - t + \displaystyle\int_{t+y(t)-\exp(-t)-\ln(2)}^{t} \exp(s)y(s)ds, & 0 \leq t \leq 1 \\ y(t) = 0, & t < 0 \end{cases} \tag{5.3}$$

Table 5 The errors $|Z_1 - Z_{h1}|$ and $\|y - u\|_\infty^{[0,1]}$ in Example 3

h	$\|Z_1 - Z_{h1}\|$		$\\|y - u\\|_\infty^{[0,1]}$	
	$p = 3$	$p = 4$	$p = 3$	$p = 4$
h_0	7.74E−06	2.67E−07	3.28E−04	4.47E−06
$h_0/2$	5.26E−07	1.86E−08	4.26E−05	2.98E−07
$h_0/4$	1.96E−08	1.20E−09	5.48E−06	1.94E−08
$h_0/8$	1.08E−09	7.86E−11	6.94E−07	1.23E−09
$h_0/16$	6.70E−11	4.98E−12	8.73E−08	7.76E−11

Table 6 Convergence orders of $|Z_1 - Z_{h1}|$ and $\|y-u\|_\infty^{[0,1]}$ in Example 3

h	$\|Z_1 - Z_{h1}\|$		$\|y-u\|_\infty^{[0,1]}$	
	$p=3$	$p=4$	$p=3$	$p=4$
h_0	3.84	3.84	2.94	3.90
$h_0/2$	4.75	3.96	2.96	3.95
$h_0/4$	4.18	3.93	2.98	3.98
$h_0/8$	4.00	3.98	2.99	3.99
$h_0/16$	3.99	3.98	3.00	3.99

with the exact solution

$$y(t) = \begin{cases} \exp(-t) & 0 \le t \le \ln(2) \\ \frac{1}{2}\mu(t)\left[2\exp\left(-t-\frac{1}{2}\exp(t)\right) + E_1(1) + E_1\left(\frac{1}{2}\exp(t)\right)\right], & \ln(2) \le t \le 1 \end{cases}$$

where $\mu(t) = \exp\left(-\frac{1}{2}\exp(t)\right)$ and

$$E_1(x) = \int_1^\infty \frac{e^{-xt}}{t} dt = \Gamma(0,x), \qquad x > 0,$$

such that $\Gamma(0,x)$ is the generalized Gamma function [26]. Here, the collocation parameters c_i's are considered as Gauss Lobatto points, i.e. the zeros of $s(s-1)P'_{m-1}(2s-1)$, where $P_{m-1}(s)$ is the Legendre polynomial of degree $m-1$ (see e.g. [6]), also $Z_1 = \ln(2) \approx 0.6931472$. The maximum error $\|y-u\|_\infty^{[0,1]}$ and the absolute error $|Z_1 - Z_{h1}|$ are tabulated for different values of p and h with $h_0 = 0.25$ in Tables 5 and 6.

Figures 5 and 6 show the point-wise error behaviors $|y(t) - u(t)|$ in the collocation points $t_{n,i}$, for different values of h and $p = 3,\ 4$. As can be seen, there is a jump at the singular point $Z_1 \approx 0.6931472$.

Example 4 (From [27]) The following problem has been considered as an optimal replacement model in mathematical economics in [27]:

Table 7 The obtained numerical results in Example 4. with different values of h

h	$\|y-u\|_\infty^{[1,2]}$		Order of convergence	
	$p=2$	$p=4$	$p=2$	$p=4$
h_0	7.55E−05	1.72E−12	1.93	3.86
$h_0/2$	1.98E−05	1.19E−13	1.99	4.04
$h_0/4$	4.97E−06	7.19E−15	1.98	3.96
$h_0/8$	1.26E−06	4.62E−16	2.00	4.01
$h_0/16$	3.15E−07	2.86E−17	2.00	4.01
$h_0/32$	7.89E−08	1.78E−18	2.00	4.00

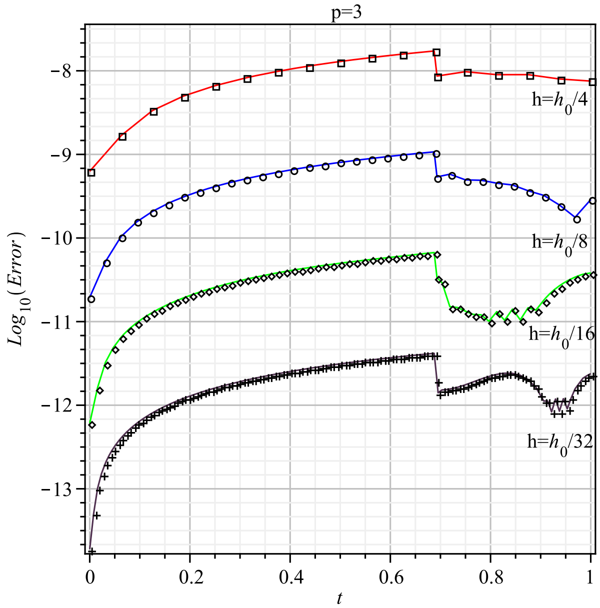

Fig. 5 Point-wise absolute errors for $p = 3$, and different values of h in Example 3

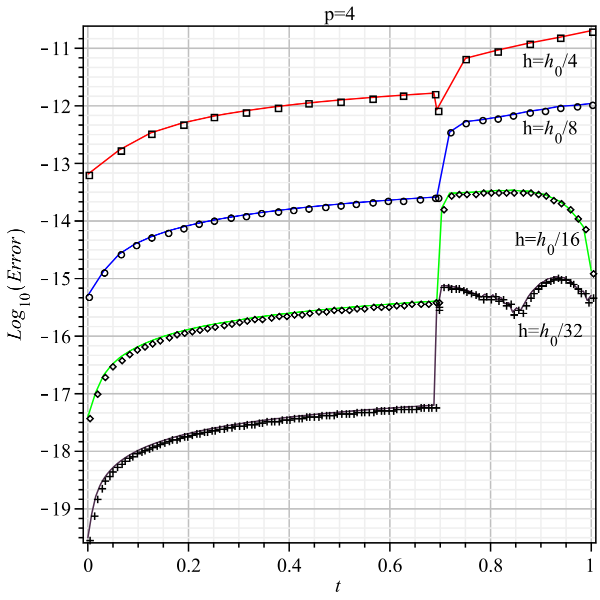

Fig. 6 Point-wise absolute errors for $p = 4$, and different values of h in Example 3

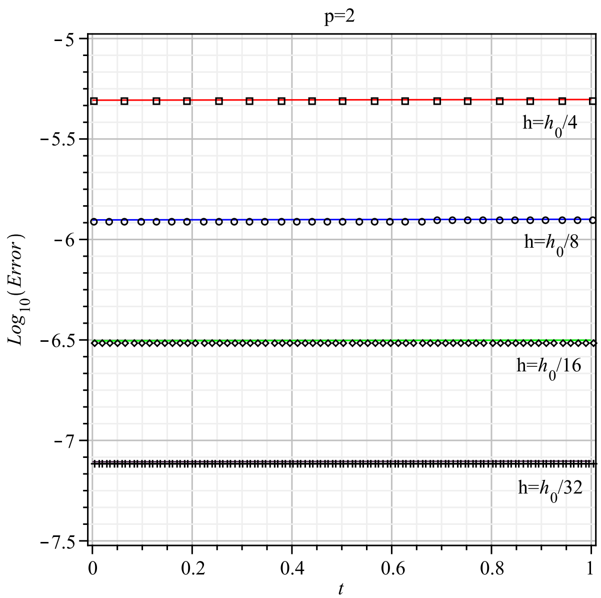

Fig. 7 Point-wise absolute errors for $p = 2$, and different values of h in Example 4

$$\begin{cases} \displaystyle\int_t^{y^{-1}(t)} e^{-r\tau}[\beta(t,\tau) - \beta(y(\tau),\tau)]d\tau = e^{-rt}\gamma(t), & t \in [1,\infty) \\ y(t) = 1 - \sqrt{2}, & t \in [0,1) \end{cases} \tag{5.4}$$

where $y^{-1}(t)$ is the inverse of $y(t)$ as a solution and the constant $r > 0$ is the industry-wide discount rate. The unknown function $y(t)$ in the integrand and the upper limit of integration essentially complicates the analysis of this problem.

We work with the same conditions as [27]. For instance, for the restricted case $\beta(t,\tau) \equiv \beta(\tau) = \tau$, $\gamma(t) = 1$ and $r = 0.01$, the solution of (5.4) is $y(t) = t - A$, where $A \approx \sqrt{2}$ and $t \in [1,\infty)$. Following [21, 27], we also focus on constructing the solutions $y(t)$ of (5.4) for some bounded interval $[t_0, T]$ such that $y(t) < t$. However, the condition $T \gg t_0$, may be considered as an applied problem which has investigated theoretically in [27] and numerically as well in [21]. Here, $Z_0 = t_0 = 1$ and $z_0 = 1$.

Because of the solution is smooth at the initial point $t = 1$, propagation of the singular points can not be seen. Our numerical experiments give relatively satisfactory errors when compared with the solution [27]. In Table 7, we have presented the overall maximum errors and order of convergence for different values of h with $h_0 = 0.25$ and $p = 2,\ 4$.

Figures 7 and 8 show the point-wise absolute errors $|y(t) - u(t)|$ in the collocation points $t_{n,i}$, for $p = 2,\ 4$ and different values of h.

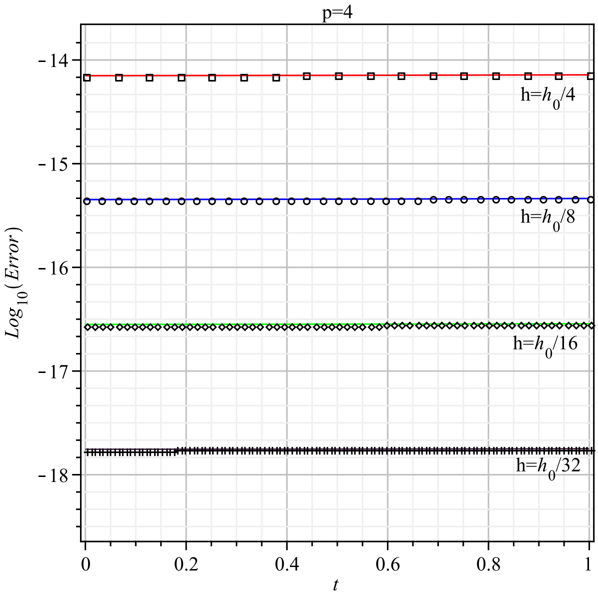

Fig. 8 Point-wise absolute errors for $p = 4$, and different values of h in Example 4

6 Conclusion

This work is concerned with the extension of the piecewise collocation method to a class of state-dependent delay integral equations. Since obtaining the singular points of this equations involves a nonlinear equation to be solved, the dense-output solutions (such as piecewise collocation) are suitable for their numerical analysis. We analyzed the convergence properties of the method and included the propagated discontinuities in the set of the mesh points. We have implemented our approach as an experimental Maple® code and carried out numerical experiments over some test problems. One of the possible extensions of the method given here is to investigate the approximate solution of state-dependent delay intgero-differential equations.

Acknowledgments The authors would like to thank the two anonymous referees for their careful reading of the manuscript and their valuable comments and suggestions. Deepest gratitude are also due to Dr. Saeed Pishbin for the help with computational codes and relevant discussions.

References

1. Baker, C.T.H., Paul, C.A.H., Wille, D.R.: Issues in the numerical solution of evolutionary delay differential equations. J. Adv. Comput. Math. **3**(3), 171–196 (1995)
2. Bélair, J.: Population models with state-dependent delays. In: Lecture Notes in Pure and Applied Mathematics, vol. 131., Dekker, New York (1991)

3. Bélair, J., Mackey, M.C.: Consumer memory and price fluctuations on commodity markets: an integro differential model. J. Dynam. Differ. Equ. **1**(3), 299–325 (1989)
4. Bellen, A., Zennaro, M.: Numerical Methods for Delay Differential Equations. Oxford University Press (2003)
5. Bellman, R.: On the computational solution of differential-difference equations. J. Math. Anal. Appl. **2**, 108–110 (1961)
6. Brunner, H.: Collocation Methods for Volterra Integral and Related Functional Equations. Cambridge University Press (2004)
7. Brunner, H.: The numerical analysis of functional integral and integro-differential equations of Volterra type. Acta Numerica **13**, 55–145 (2004)
8. Cahlon, B., Nachman, L.J.: Numerical solutions of Volterra integral equations with a solution dependent delay. J. Math. Anal. Appl. **112**(2), 541–562 (1985)
9. Cahlon, B.: Numerical solutions for functional integral equations with state-dependent delay. Appl. Numer. Math. **9**(3-5), 291–305 (1992)
10. Driver, R.D.: Existence theory for a delay-differential system. Contrib. Differ. Equa. **1**, 317–336 (1963)
11. Feldstein, A., Neves, K.W.: High order methods for state-dependent delay differential equations with nonsmooth solutions. SIAM J. Numer. Anal. **21**(5), 844–863 (1984)
12. Feldstein, A., Neves, K.W., Thompson, S.: Sharpness results for state-dependent delay differential equations: an overview. Appl. Numer. Math. **56**(3-4), 472–487 (2006)
13. Guglielmi, N., Hairer, E.: Implementing Radau-IIA methods for stiff delay differential equations. Computing **67**(1), 1–12 (2001)
14. Guglielmi, N., Hairer, E.: Computing breaking points in implicit delay differential equations. Adv. Comput. Math. **29**(3), 229–247 (2008)
15. Györi, I., Hartung, F., Turi, J.: Numerical approximations for a class of differential equations with time and state-dependent delays. Appl. Math. Lett. **8**, 19–24 (1995)
16. Hartung, F., Turi, J.: On the asymptotic behavior of the solutions of a state dependent differential equation. Diff. Integr. Eqns. **8**, 1867–1872 (1995)
17. Hartung, F., Krisztin, T., Walther, H.O.: Functional differential equations with state-dependent delays: theory and applications. In: Cañada, A., Drabek, P., Fonda, A. (eds.) Handbook of Differential Equations: Ordinary Differential Equations, p. 3 (2006)
18. Hauber, R.: Numerical treatment of retarted differential-algebraic equations by collocation methods. Adv. Comput. Math. **7**(4), 573–592 (1997)
19. Hbid, M.L., Sánchez, E., Bravo de la Parra, R.: State-dependent delays asssociated to threshold phenomena in structured population dynamics. Math. Model. Methods Appl. Sci. **17**(6), 877–900 (2007)
20. Hbid, M.L., Louihi, M., Sánchez, E.: A threshold state-dependent delayed functional equation arising from marine population dynamics: modelling and analysis. J. Evol. Equ. **10**(4), 905–928 (2010)
21. Hritonenko, N., Yatsenko, Y.: Integral equation of optimal replacement: analysis and algorithms. Appl. Math. Model. **33**, 2737–2747 (2009)
22. Insperger, T., Stépan, G., Turi, J.: State-dependent delay in regenerative turning processes. Nonlinear. Dyn. **47**(2-3), 275–283 (2007)
23. Milgram, M.S.: The generalized integro-exponential function. Math. Comp. **44**, 443–458 (1985)
24. Neves, K.W., Feldstein, A.: Characterization of jump discontinuities for state-dependent delay integral equations. J. Math. Anal. Appl. **56**(3), 689–707 (1976)
25. Shampine, L.F.: Solving ODEs and DDEs with residual control. Appl. Numer. Math. **52**, 113–127 (2005)
26. Walther, H.O.: Stable periodic motion of a system with state-dependent delay. Diff. Integr. Eqns. **15**(8), 923–944 (2002)
27. Yatsenko, Y.: Volterra integral equations with unknown delya time. Method Appl. Anal. **2**, 408–419 (1995)
28. ZivariPiran, H., Enright, W.H.: An efficient unified approach for the numerical solution of delay differential equations. Numer. Algor. **53**(2-3), 397–417 (2010)