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ON THE CONVERGENCE ANALYSIS OF THE SPLINE COLLOCATION METHOD FOR SYSTEM OF INTEGRAL ALGEBRAIC EQUATIONS OF INDEX-2

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This article presents some theoretical results for polynomial spline collocation solution to a new class of semi-explicit Integral Algebraic Equations (**IAEs**) of index-2, which has been introduced in a recent paper of the authors Hadizadeh, M., Ghoreishi, F. and Pishbin, S. [2011] "Jacobi spectral solution for integral-algebraic equations of index-2," *Appl. Numer. Math.* **61**, 131–148. Critical issues for numerical analysis of the spline collocation method for this type of Volterra systems are discussed and the necessary and sufficient conditions are presented which guarantee the convergence of the method. We analyze the rate of convergence for two disjoint cases of collocation parameter c_m. Numerical results confirm the rate of decay of the error predicted by this theory.

Keywords: Integral algebraic equation; system of Volterra integral equation; index of IAEs; spline collocation method; error analysis; numerical treatment.

AMS Subject Classification: 65R20, 45F15.

1. Introduction

The principal aim of this paper is to serve the numerical analysis of the spline collocation method for a mixed system of Volterra integral equations of the first and second kind which is known as Integral Algebraic Equations (**IAEs**). In fact, the present work is concerned with the semi-explicit system of integral algebraic equations of index-2

$$\begin{cases} y(t) = f(t) + (\nu_{11}y)(t) + (\nu_{12}z)(t), \\ 0 = g(t) + (\nu_{21}y)(t), \end{cases} \tag{1}$$

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where the Volterra integral operators ν_{kl} are given by,

$$(\nu_{kl}\varphi)(t) = \int_0^t K_{kl}(t,s)\varphi(s)ds, \quad t \in I = [0,T], \quad (k,l = 1,2),$$

and $y, f : I \to \mathbb{R}^{d_1}$, $z, g : I \to \mathbb{R}^{d_2}$, $K_{kk}(.,.) \in L(\mathbb{R}^{d_k})$, $K_{12}(.,.) \in L(\mathbb{R}^{d_2}, \mathbb{R}^{d_1})$, $K_{21}(.,.) \in L(\mathbb{R}^{d_1}, \mathbb{R}^{d_2})$ are continuous functions and $L(.,.)$ is the linear transformation space. The system (1) is a special case of the following general form of IAEs

$$A(t)X(t) = G(t) + \int_0^t K(t,s,X(s))ds, \quad t \in I = [0,T],$$

where $A(t)$ is a singular matrix with continuous entries (rank(A)≥ 1, $\det(A) = 0$) and $G(t)$ is a known free term.

Such equations and systems frequently arise in many physical and applied problems especially in the fields of viscoelastic materials [Janno and Wolfersdorf (1997)], dynamic processes in chemical reactors [Kafarov *et al.* (1999)], identification of memory kernels in heat conduction [Wolfersdorf (1994)], evolution of a chemical reaction within a small cell [Jumarhon *et al.* (1996)] and the two dimensional biharmonic equation in a semi-infinite strip [Gomilko (2003)]. A primary and general theory of IAEs are due to Chistyakov and Gear in 1986–1990, who determined the theory and difficulties of these equations [see, for instance Chistyakov (1986) and Gear (1990)]. A system of IAEs is assigned by a number which is known as index, to measure its complexity concerning both theoretical and numerical treatments. Actually, the index plays a key role in the solvability and regularity of the solution of IAEs.

There has been few work on analyzing and numerical methods for the IAEs in literature. The existence and uniqueness results of solution for IAEs systems with convolutions kernels have been discussed in Bulatov and Chistyakov [1997]. Kauthen [2000], applied the polynomial spline collocation method for a semi-explicit IAEs with index-1 and established global convergence as well as local superconvergence. Brunner [2004] defined the index-1 tractable for a semi-explicit form of IAEs and investigated the existence of a unique solution for this type of systems. Recently, the authors in Hadizadeh *et al.* [2011] have defined the index-2 tractable for a new class of IAEs and proposed a Jacobi collocation method including the matrix-vector multiplication representation of the equation. Most recently, the numerical analysis of the two-dimensional integral-algebraic system has also investigated by Bulatov and Lima in [2011].

As we mentioned, the solution of IAEs has been closely related to the definition of index. Different definitions of index have been given in the literature, see for instance Gear [1990], Brunner [2004], and Hadizadeh *et al.* [2011]. The "index reduction procedure" for IAEs systems has been introduced by Gear [1990], in which if the process is terminated, then the index is determined. This means that under suitable conditions, there is a solution for the resulting regular system of integral equations. Here, our mean about the index for the IAEs system (1), is the minimum number of differentiation of the second equation of (1) with respect to t, in order a system of regular VIEs has been reached. Generally, this methodology has several restrictions that can causes instability from a numerical point of view. Since the smoothness condition of the second equation of (1) is not often satisfied in some applications, we may consider the index notion without use of derivative arrays (e.g., the tractability index which is due to Brunner [2004] and the authors Hadizadeh *et al.* [2011]). Also, it is worth mentioning that the numerical schemes which are applicable (i.e., convergent) for IAEs of a given index, might not be useful for IAEs of higher index. Note that the IAEs with index > 1 are generally hard to solve and are still under active research.

Throughout this paper, we consider the case when the conditions of Theorem 1 in Hadizadeh *et al.* [2011] be satisfied. Hence, in virtue of the smoothness of the given functions, the existence, uniqueness and the regularity results for continuous solutions of the IAEs system (1) may be established from the following theorem:

Theorem 1. (From Hadizadeh *et al.* [2011]) *Let* $\nu \geq 0$ *and assume that*

(1) $K_{1l} \in C^{\nu}(D)$ *for* $l = 1, 2$ *and* $D = \{(t,s) :\ 0 \leq s \leq t \leq T\}$,
(2) $K_{21} \in C^{\nu+1}(D)$ *and* $|\det(K_{21}(t,t)K_{12}(t,t))| \geq k_0 > 0$,
(3) $f \in C^{\nu}(D),\ g \in C^{\nu+1}(D)$ *and* $g_1(0) = 0$,

then the IAEs (1) *possesses a unique solution* $y, z \in C^{\nu}(I)$.

The main aim of this work is to provide the necessary and sufficient conditions for convergence of the spline collocation method for the IAEs system (1). For this purpose, we first apply the polynomial spline collocation method for the numerical solvability of the index-2 system of IAEs (1) in Sec. 2. A critical global convergence analysis of the method for two disjoint cases of collocation parameter c_m is then given in Sec. 3 and finally in Sec. 4, some numerical experiments are reported which supporting the theoretical results.

2. The Spline Collocation Method

Let $\Pi_N = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ be a uniform partition of the interval $I = [0,T]$ with grid points $t_n = nh,\ (n = 0, \ldots, N)$. Also, let h be the stepsize and Θ is given by:

$$\Theta = \{t_{nj} = t_n + c_j h :\ 0 < c_1 < c_2 < \cdots < c_m \leq 1, 0 \leq n \leq N-1\},$$

where $c_j\ (j = 1, \ldots, m)$ and t_{nj} are the collocation parameters and the collocation points, respectively. We define the subintervals $\sigma_n = (t_n, t_{n+1}],\ n = 0, \ldots, N-1$, and the space of piecewise polynomials of degree $m - 1 \geq 0$, as follows:

$$S_{m-1}^{-1}(\Pi_N) = \{\upsilon : \upsilon|_{\sigma_n} \in \Pi_{m-1}, (0 \leq n \leq N-1)\}.$$

Consider the IAEs system (1) and approximate the solution $(y_1, \ldots, y_{d_1}, z_1, \ldots, z_{d_2})^T$ by $(\mathbf{u}, \mathbf{v})^T$, where $\mathbf{u} = \{u_l\}_{l=1}^{d_1}$, $\mathbf{v} = \{v_l\}_{l=1}^{d_2}$ and each component of $\mathbf{u}$ and $\mathbf{v}$ belongs to $S_{m-1}^{-1}(\Pi_N)$. The collocation solution $\mathbf{u}, \mathbf{v}$ can be defined by the following collocation equations

$$\mathbf{u}(t) = \mathbf{f}(t) + \int_0^t \mathbf{K}_{11}(t,s)\mathbf{u}(s)ds + \int_0^t \mathbf{K}_{12}(t,s)\mathbf{v}(s)ds, \tag{2}$$

$$\mathbf{0} = \mathbf{g}(t) + \int_0^t \mathbf{K}_{21}(t,s)\mathbf{u}(s)ds, \tag{3}$$

where $\mathbf{f}(t) = \{f_l(t)\}_{l=1}^{d_1}$, $\mathbf{g}(t) = \{g_l(t)\}_{l=1}^{d_2}$, $\mathbf{K}_{pq}(t,s) = \{K_{pq}(t,s)\}_{d_p \times d_q}$, $(p, q = 1, 2)$, $\mathbf{0} = \{0\}_{d_2 \times 1}$ and $t \in \Theta$.

Since $\{u_l\}_{l=1}^{d_1}, \{v_l\}_{l=1}^{d_2} \in S_{m-1}^{-1}(\Pi_N)$, (for $\rho \in (0,1]$), the following relations hold:

$$\mathbf{u}(t_n + \rho h) = \sum_{j=1}^{m} \mathbf{U}_{nj} L_j(\rho), \quad \mathbf{U}_{nj} = \mathbf{u}(t_n + c_j h), \tag{4}$$

$$\mathbf{v}(t_n + \rho h) = \sum_{j=1}^{m} \mathbf{V}_{nj} L_j(\rho), \quad \mathbf{V}_{nj} = \mathbf{v}(t_n + c_j h), \tag{5}$$

where $\mathbf{U}_{nj} = \{U_{lnj}\}_{l=1}^{d_1}, \mathbf{V}_{nj} = \{V_{lnj}\}_{l=1}^{d_2}$ and $L_j(\rho)$ represents the Lagrange canonical polynomials for the collocation parameters $\{c_j\}$ which is defined as,

$$L_j(\rho) = \prod_{k \neq j} \frac{(\rho - c_k)}{(c_j - c_k)}, \quad j = 1, \ldots, m.$$

Let us set $\rho = (s - t_i)/h$, $(i = 0, \ldots, n)$, and insert (4) and (5) into relations (2) and (3), respectively, then we have

$$\begin{aligned} \mathbf{U}_{nj} = \mathbf{f}(t_{nj}) &+ h \sum_{i=0}^{n-1} \sum_{k=1}^{m} \left(\int_0^1 \mathbf{K}_{11}(t_{nj}, t_i + \rho h) \mathbf{U}_{ij} L_k(\rho) d\rho \right) \\ &+ h \sum_{i=0}^{n-1} \sum_{k=1}^{m} \left(\int_0^1 \mathbf{K}_{12}(t_{nj}, t_i + \rho h) \mathbf{V}_{ij} L_k(\rho) d\rho \right) \\ &+ h \sum_{k=1}^{m} \left(\int_0^{c_j} \mathbf{K}_{11}(t_{nj}, t_n + \rho h) \mathbf{U}_{nj} L_k(\rho) d\rho \right) \\ &+ h \sum_{k=1}^{m} \left(\int_0^{c_j} \mathbf{K}_{12}(t_{nj}, t_n + \rho h) \mathbf{V}_{nj} L_k(\rho) d\rho \right), \end{aligned} \tag{6}$$

$$\begin{aligned} \mathbf{0} = \mathbf{g}(t_{nj}) &+ h \sum_{i=0}^{n-1} \sum_{k=1}^{m} \left(\int_0^1 \mathbf{K}_{21}(t_{nj}, t_i + \rho h) \mathbf{U}_{ij} L_k(\rho) d\rho \right) \\ &+ h \sum_{k=1}^{m} \left(\int_0^{c_j} \mathbf{K}_{21}(t_{nj}, t_n + \rho h) \mathbf{U}_{nj} L_k(\rho) d\rho \right). \end{aligned} \tag{7}$$

Note that, the integrals in (6) and (7) can not be computed explicitly in general, and thus one has to use appropriate quadrature rules to approximate them,

$$\int_0^{c_j} \mathbf{K}_{pq}(t_{nj}, t_n + \rho h) L_i(\rho) d\rho \approx \sum_{k=1}^{m} \mathbf{K}_{pq}(t_{nj}, t_n + c_k h) a_{jk},$$

$$\int_0^{1} \mathbf{K}_{pq}(t_{nj}, t_n + \rho h) L_i(\rho) d\rho \approx \sum_{k=1}^{m} \mathbf{K}_{pq}(t_{nj}, t_n + c_k h) b_{k},$$

where $p, q = 1, 2$, and the coefficients a_{jk} and b_k are defined by:

$$a_{jk} = \int_0^{c_j} L_k(\rho) d\rho, \quad b_k = \int_0^1 L_k(\rho) d\rho, \quad j, k = 1, \ldots, m.$$

By inserting the above relations into the equations (6) and (7), we get

$$\begin{aligned} \mathbf{U}_{nj} = \mathbf{f}(t_{nj}) &+ h \sum_{i=0}^{n-1} \sum_{k=1}^{m} \mathbf{K}_{11}(t_{nj}, t_{ik}) \mathbf{U}_{ik} b_k + h \sum_{i=0}^{n-1} \sum_{k=1}^{m} \mathbf{K}_{12}(t_{nj}, t_{ik}) \mathbf{V}_{ik} b_k \\ &+ h \sum_{k=1}^{m} \mathbf{K}_{11}(t_{nj}, t_{nk}) \mathbf{U}_{nk} a_{jk} + h \sum_{k=1}^{m} \mathbf{K}_{12}(t_{nj}, t_{nk}) \mathbf{V}_{nk} a_{jk}, \end{aligned}$$

$$\mathbf{0} = \mathbf{g}(t_{nj}) + h \sum_{i=0}^{n-1} \sum_{k=1}^{m} \mathbf{K}_{21}(t_{nj}, t_{ik}) \mathbf{U}_{ik} b_k + h \sum_{k=1}^{m} \mathbf{K}_{21}(t_{nj}, t_{nk}) \mathbf{U}_{nk} a_{jk}.$$

Now, by substituting $\mathbf{U}_{nj}$ and $\mathbf{V}_{nj}$ as solution of the resulting system into (4) and (5), we can get the numerical solution of IAEs system (1) for arbitrary $\rho \in (0, 1]$.

3. Convergence Analysis

In this section, we give the main result of this paper in order to preserve the global convergence of the proposed scheme. The analysis employed here is essentially relying on the given approach of the polynomial spline collocation method for the system of IAEs of index-1 in Kauthen [2000] including some innovations and new initiatives in the case $c_m < 1$, which lead to completely different order of convergence results.

Theorem 2. *Assume that* $\mathbf{u} = (u_1, \ldots, u_{d_1})^T$ *and* $\mathbf{v} = (v_1, \ldots, v_{d_2})^T$ *with* $\{u_l\}_{l=1}^{d_1}, \{v_l\}_{l=1}^{d_2} \in S_{m-1}^{-1}(\Pi_N)$ *are the collocation approximations to the solution* $\mathbf{y} = (y_1, y_2, \ldots, y_{d_1})^T$, $\mathbf{z} = (z_1, z_2, \ldots, z_{d_2})^T$ *of the index-2 IAEs system* (1) *which is defined by* (4) *and* (5) *and let the hypotheses of Theorem* 1 *hold. For the collocation parameter* c_m, *two following cases can be considered*:

(1) *If* $c_m = 1$, *the collocation approximation* $(\mathbf{u}, \mathbf{v})^T$ *converges to the solution* $(\mathbf{y}, \mathbf{z})^T$, *and we have*

$$\|\mathbf{y} - \mathbf{u}\|_\infty = O(h^m), \quad \|\mathbf{z} - \mathbf{v}\|_\infty = O(h^{m-1}).$$

(2) *If* $c_m < 1$, *the collocation approximation* $(\mathbf{u},\mathbf{v})^T$ *converges to the solution* $(\mathbf{y},\mathbf{z})^T$ *for any* $m \geq 3$, *if and only if*

$$-1 \leq \rho_m = (-1)^m \prod_{i=1}^{m} \frac{1-c_i}{c_i} \leq 1.$$

Furthermore, the following order of convergence holds:

$$\|\mathbf{y}-\mathbf{u}\|_\infty = \begin{cases} O(h^m), & if\ \rho_m \in [-1,1), \\ O(h^{m-1}), & if\ \rho_m = 1, \end{cases}$$

$$\|\mathbf{z}-\mathbf{v}\|_\infty = \begin{cases} O(h^{m-1}), & if\ \rho_m \in (-1,1), \\ O(h^{m-2}), & if\ \rho_m = -1, \\ O(h^{m-3}), & if\ \rho_m = 1, \end{cases}$$

1 *as* $h \to 0$, *with* $Nh \leq const$.

Proof. Using Peano's Theorem and Corollary (1.8.2) from Brunner [2004], pp. 43 and 44, the exact solution of the IAEs system (1) can be obtained as:

$$\mathbf{y}(t_n + \rho h) = \sum_{k=1}^{m} \mathbf{Y}_{nk} L_k(\rho) + h^m \mathbf{R}_{m,n}(\rho), \tag{8}$$

$$\mathbf{z}(t_n + \rho h) = \sum_{k=1}^{m} \mathbf{Z}_{nk} L_k(\rho) + h^m \mathbf{R}'_{m,n}(\rho), \tag{9}$$

where $\mathbf{Y}_{nk} = \{Y_{lnk}\}_{l=1}^{d_1} = \mathbf{y}(t_n + c_k h)$, $\mathbf{Z}_{nk} = \{Z_{lnk}\}_{l=1}^{d_2} = \mathbf{z}(t_n + c_k h)$, and the Peano remainder and kernel are given by

$$\mathbf{R}_{m,n}(\rho) = \int_0^1 \mathbf{y}^{(m)}(t_n + xh) K_m(\rho, x)dx,$$

$$\mathbf{R}'_{m,n}(\rho) = \int_0^1 \mathbf{z}^{(m)}(t_n + xh) K_m(\rho, x)dx,$$

with

$$K_m(\rho, x) = \frac{1}{(m-1)} \left\{ (\rho - x)_+^{(m-1)} - \sum_{k=1}^{m} L_k(\rho)(c_k - x)_+^{(m-1)} \right\}, \quad \rho \in (0,1].$$

Suppose that $\mathbf{e} = \mathbf{y} - \mathbf{u}$ and $\mathbf{e}' = \mathbf{z} - \mathbf{v}$ are the collocation errors, then by considering the relations (8), (4) and (9), (5), we have the following representations

$$\mathbf{e}(t_n + \rho h) = \sum_{k=1}^{m} \mathbf{E}_{nk} L_k(\rho) + h^m \mathbf{R}_{m,n}(\rho), \tag{10}$$

$$\mathbf{e}'(t_n + \rho h) = \sum_{k=1}^{m} \mathbf{E}'_{nk} L_k(\rho) + h^m \mathbf{R}'_{m,n}(\rho), \tag{11}$$

2 where $\mathbf{E}_{nk} = \mathbf{Y}_{nk} - \mathbf{U}_{nk}$ and $\mathbf{E}'_{nk} = \mathbf{Z}_{nk} - \mathbf{V}_{nk}$.

Subtracting the first and second equation of (1) from (2) and (3), respectively, we obtain

$$\mathbf{e}(t) = \int_0^t \mathbf{K}_{11}(t,s)\mathbf{e}(s)ds + \int_0^t \mathbf{K}_{12}(t,s)\mathbf{e}'(s)ds, \tag{12}$$

$$\mathbf{0} = \int_0^t \mathbf{K}_{21}(t,s)\mathbf{e}(s)ds. \tag{13}$$

Let us set $\rho = (s - t_i)/h,\ (i = 0, \ldots, n)$, then we will have

$$\begin{aligned}
\mathbf{e}(t_{nj}) = h\sum_{i=0}^{n-1}\int_0^1 \mathbf{K}_{11}(t_{nj,t_i+\rho h)}\mathbf{e}(t_i+\rho h)d\rho \\
+ h\sum_{i=0}^{n-1}\int_0^1 \mathbf{K}_{12}(t_{nj}, t_i+\rho h)\mathbf{e}'(t_i+\rho h)d\rho \\
+ h\int_0^{c_j} \mathbf{K}_{11}(t_{nj}, t_n+\rho h)\mathbf{e}(t_n+\rho h)d\rho \\
+ h\int_0^{c_j} \mathbf{K}_{12}(t_{nj}, t_n+\rho h)\mathbf{e}'(t_n+\rho h)d\rho,
\end{aligned} \tag{14}$$

$$\begin{aligned}
\mathbf{0} = h\sum_{i=0}^{n-1}\int_0^1 \mathbf{K}_{21}(t_{nj}, t_i+\rho h)\mathbf{e}(t_i+\rho h)d\rho \\
+ h\int_0^{c_j} \mathbf{K}_{21}(t_{nj}, t_n+\rho h)\mathbf{e}(t_n+\rho h)d\rho.
\end{aligned} \tag{15}$$

We now rewrite (15) with n replaced by $n-1$ and $j = m$, subtract this equation from (15) and divide by h:

$$\begin{aligned}
&\int_0^{c_j} \mathbf{K}_{21}(t_{nj}, t_n+\rho h)\mathbf{e}(t_n+\rho h)d\rho \\
&\quad = \int_0^{c_m} \mathbf{K}_{21}(t_{n-1,m}, t_{n-1}+\rho h)\mathbf{e}(t_{n-1}+\rho h)d\rho \\
&\qquad - \int_0^1 \mathbf{K}_{21}(t_{nj}, t_{n-1}+\rho h)\mathbf{e}(t_{n-1}+\rho h)d\rho \\
&\qquad - \sum_{i=0}^{n-2}\int_0^1 \mathbf{K}_{21}(t_{nj}, t_i+\rho h) - \mathbf{K}_{21}(t_{n-1,m}, t_i+\rho h)\mathbf{e}(t_i+\rho h)d\rho.
\end{aligned} \tag{16}$$

In this position, we employ the *Taylor* series expansion of the components of $\mathbf{K}_{21}$ as follows:

$$\begin{aligned}
&\mathbf{K}_{21}(t_{nj}, t_i+\rho h) - \mathbf{K}_{21}(t_{n-1,m}, t_i+\rho h) \\
&\quad = c_j h\frac{\partial \mathbf{K}_{21}}{\partial t}(t_n, t_i+\rho h) + (1-c_m)h\frac{\partial \mathbf{K}_{21}}{\partial t}(t_n, t_i+\rho h) + \mathbf{O}(h).
\end{aligned} \tag{17}$$

1 where $\mathbf{O}(h) = \{O(h)\}_{d_2\times d_1}$.

We have to distinguish between two cases for c_m, so let us assume that the domain of c_m be divided into two disjoint subsets, $c_m < 1$ and $c_m = 1$. We analyze these two cases separately as follows:

Case 1: $c_m = 1$.

Inserting (17) into (16), and considering the *Taylor* series expansion of $(\mathbf{K}_{21}(t_{n-1,m}, t_{n-1}+\rho h) - \mathbf{K}_{21}(t_{nj}, t_{n-1}+\rho h))$, similar to (17), we can rewrite (16) as

$$\int_0^{c_j} \mathbf{K}_{21}(t_{nj}, t_n + \rho h)\mathbf{e}(t_n + \rho h)d\rho = -\sum_{i=0}^{n-1} \int_0^1 c_j h \frac{\partial \mathbf{K}_{21}}{\partial t}(t_n, t_i + \rho h)\mathbf{e}(t_i + \rho h)d\rho. \tag{18}$$

Substituting (10) into (18), yields

$$\begin{aligned} \mathbf{0} = &-\sum_{k=1}^{m} \int_0^{c_j} \mathbf{K}_{21}(t_{nj}, t_n + \rho h)\mathbf{E}_{nk} L_k(\rho)d\rho \\ &- \sum_{i=0}^{n-1}\sum_{k=1}^{m} \int_0^1 c_j h \frac{\partial \mathbf{K}_{21}}{\partial t}(t_n, t_i + \rho h)\mathbf{E}_{ik} L_k(\rho)d\rho \\ &- h^m \int_0^{c_j} \mathbf{K}_{21}(t_{nj}, t_n + \rho h)\mathbf{R}_{m,n}(\rho)d\rho \\ &- h^m \sum_{i=0}^{n-1} \int_0^1 c_j h \frac{\partial \mathbf{K}_{21}}{\partial t}(t_n, t_i + \rho h)\mathbf{R}_{m,n}(\rho)d\rho. \end{aligned} \tag{19}$$

We also rewrite (14), using (11) and (10) as

$$\begin{aligned} \mathbf{E}(t_{nj}) = h &\sum_{i=0}^{n-1}\sum_{k=1}^{m} \int_0^1 \mathbf{K}_{11}(t_{nj}, t_i + \rho h)\mathbf{E}_{ik} L_k(\rho)d\rho \\ &+ h \sum_{i=0}^{n-1}\sum_{k=1}^{m} \int_0^1 \mathbf{K}_{12}(t_{nj}, t_i + \rho h)\mathbf{E}'_{ik} L_k(\rho)d\rho \\ &+ h \sum_{k=1}^{m} \int_0^{c_j} \mathbf{K}_{11}(t_{nj}, t_n + \rho h)\mathbf{E}_{nk} L_k(\rho)d\rho \\ &+ h \sum_{k=1}^{m} \int_0^{c_j} \mathbf{K}_{12}(t_{nj}, t_n + \rho h)\mathbf{E}'_{nk} L_k(\rho)d\rho \\ &+ h^{m+1} \sum_{i=0}^{n-1} \int_0^1 \mathbf{K}_{11}(t_{nj}, t_i + \rho h)\mathbf{R}_{m,n}(\rho)d\rho \\ &+ h^{m+1} \sum_{i=0}^{n-1} \int_0^1 \mathbf{K}_{12}(t_{nj}, t_i + \rho h)\mathbf{R}'_{m,n}(\rho)d\rho \end{aligned}$$

$$+h^{m+1}\int_0^{c_j}\mathbf{K}_{11}(t_{nj},t_n+\rho h)\mathbf{R}_{m,n}(\rho)d\rho$$
$$+h^{m+1}\int_0^{c_j}\mathbf{K}_{12}(t_{nj},t_n+\rho h)\mathbf{R}'_{m,n}(\rho)d\rho. \tag{20}$$

Note that the equations (19) and (20) can be written as a compact matrix representation:

$$\mathbf{A_1}\overline{\mathbf{E}}_n=\sum_{i=0}^{n-1}\mathbf{G}\overline{\mathbf{E}}_i+\mathbf{C}, \tag{21}$$

where

$$\overline{\mathbf{E}}_n=\begin{pmatrix}\mathbf{E}_n\\ \mathbf{E}'_n\end{pmatrix},\quad \mathbf{E}_n=\{E_{l,n}\}_{l=1}^{d_1}=(E_{ln1},\ldots,E_{lnm})^T,$$
$$\mathbf{E}'_n=\{E'_{l,n}\}_{l=1}^{d_2}=(E'_{ln1},\ldots,E'_{lnm})^T,$$

and

$$\mathbf{A_1}=\begin{pmatrix}\mathbf{I}_m-h\mathbf{B}_n^{(1,1)} & -h\mathbf{B}_n^{(1,2)}\\ \mathbf{B}_n^{(2,1)} & \{0\}_{d_2\times d_2}\end{pmatrix},$$

where $\mathbf{B}_n^{(p,l)}=\begin{pmatrix}\int_0^{c_j}\mathbf{K}_{11}(t_{nj},t_n+\rho h)L_k(\rho)d\rho\\ j,k=1,\ldots,m\end{pmatrix}$, $(p,l=1,2)$ and $\mathbf{I}_m=\mathrm{dig}(I_m)$ is $d_1\times d_1$ diagonal block matrix,

$$\mathbf{G}=\begin{pmatrix}h\mathbf{B}_{n,i}^{(1,1)} & h\mathbf{B}_{n,i}^{(1,2)}\\ -hc_j\tilde{\mathbf{B}}_{n,i}^{(2,1)} & \{0\}_{d_2\times d_2}\end{pmatrix},$$

$$\mathbf{C}=\begin{pmatrix}h^{m+1}\left(\mathbf{P}_n^{(1,1)}+\mathbf{P'}_n^{(1,2)}+\sum_{i=0}^{n-1}\mathbf{P}_{n,i}^{(1,1)}+\sum_{i=0}^{n-1}\mathbf{P'}_{n,i}^{(1,2)}\right)\\ -h^m\mathbf{P}_n^{(2,1)}-c_jh^{m+1}\sum_{i=0}^{n-1}\tilde{\mathbf{P}}_{n,i}^{(2,1)}\end{pmatrix},$$

where the meaning of the matrices $\mathbf{B}_{n,i}^{(p,l)},\tilde{\mathbf{B}}_{n,i}^{(2,1)},\mathbf{P}_n^{(1,1)},\ldots$ is clear.

Due to the assumptions of Theorem 1, we have

$$|\det(\mathbf{K}_{21}(t,t)\mathbf{K}_{12}(t,t))|>0,\quad \forall\ t\in I,$$

this shows that $\mathbf{A_1}$ is an invertible matrix with the following representation: [see Bernstein (2005), pp. 44]:

$$\mathbf{A_1^{-1}}=\begin{pmatrix}F_1^{-1}+F_1^{-1}F_2(F_4-F_3F_1^{-1}F_2)^{-1}F_3F_1^{-1} & -F_1^{-1}F_2(F_4-F_3F_1^{-1}F_2)^{-1}\\ -(F_4-F_3F_1^{-1}F_2)^{-1}F_3F_1^{-1} & (F_4-F_3F_1^{-1}F_2)^{-1}\end{pmatrix},$$

where $F_1=\mathbf{I}_m-h\mathbf{B}_n^{(1,1)},\ F_2=-h\mathbf{B}_n^{(1,2)},\ F_3=\mathbf{B}_n^{(2,1)}$ and $F_4=\{0\}_{d_2\times d_2}$.

With these notations, $\mathbf{A_1^{-1}}$ is unbounded if h is sufficiently small. Multiplying (21) by $\mathbf{A_1^{-1}}$, the block matrices $\mathbf{A_1^{-1}G}$ and $\mathbf{A_1^{-1}C}$ may be obtained as

$$\mathbf{A_1^{-1}G} = \begin{pmatrix} \{0\}_{d_1\times d_1} & \{0\}_{d_1\times d_2} \\ \mathbf{D_1} & \mathbf{D_2} \end{pmatrix}, \quad \mathbf{A_1^{-1}C} = \begin{pmatrix} \{O(h^m)\}_{d_1\times 1} \\ \{O(h^{m-1})\}_{d_2\times 1} \end{pmatrix},$$

where

$$\mathbf{D_1} = -h(F_4 - F_3F_1^{-1}F_2)^{-1}F_3F_1^{-1}\mathbf{B}_{n,i}^{(1,1)} - hc_j(F_4 - F_3F_1^{-1}F_2)^{-1}\tilde{\mathbf{B}}_{n,i}^{(2,1)},$$

$$\mathbf{D_2} = -h(F_4 - F_3F_1^{-1}F_2)^{-1}F_3F_1^{-1}\mathbf{B}_{n,i}^{(1,2)},$$

1 in which $h \to 0$, with $nh = T \leq$ const.

Considering the above representations, we conclude

$$\|\mathbf{E}_n\|_1 = O(h^m),$$

where $\|\cdot\|_1$ denotes the matrix (operator) norm induced by the L^1-norm, and also

$$\mathbf{E}'_n = \sum_{i=0}^{n-1}\mathbf{D_2}\mathbf{E}'_i + \sum_{i=0}^{n-1}\mathbf{D_1}\mathbf{E}_i + \mathbf{O}(h^{m-1}), \tag{22}$$

where $\mathbf{O}(h^{m-1}) = \{O(h^{m-1})\}_{d_2\times 1}$. So, the Eq. (22) can be written as

$$\mathbf{E}'_n = \sum_{i=0}^{n-1}\mathbf{D_2}\mathbf{E}'_i + \mathbf{O}(h^{m-1}). \tag{23}$$

It then follows from the *Gronwall's* inequality [Brunner (2004)]

$$\|\mathbf{E}'_n\|_1 = O(h^{m-1}), \tag{24}$$

therefore, the following estimates for (11) and (10) can be obtained

$$|\mathbf{e}(t_n + \rho h)| \leq \Omega_m\|\mathbf{E}_n\|_1 + h^m M_m K_m, \tag{25}$$

$$|\mathbf{e}'(t_n + \rho h)| \leq \Omega_m\|\mathbf{E}'_n\|_1 + h^m M'_m K_m, \tag{26}$$

where

$$M_m = \|\mathbf{y}^{(m)}\|_\infty, \quad M'_m = \|\mathbf{z}^{(m)}\|_\infty, \quad K_m = \max_{\rho\in(0,1]}\int_0^1 |K_m(\rho, x)|dx,$$

and

$$\Omega_m = \max_j \|L_j\|_\infty.$$

Now, using (25) and (26) for $c_m = 1$, we get

$$\|\mathbf{y} - \mathbf{u}\|_\infty = O(h^m), \quad \|\mathbf{z} - \mathbf{v}\|_\infty = O(h^{m-1}).$$

2 **Case 2:** $c_m < 1$.

In this case, we will make use of the result of Brunner [2004] which gives the conditions and global order of convergence of the spline collocation method for the first kind integral equations. Actually, as stated in Theorem 2.4.2 of Brunner [2004], pp. 123, we note that the second equation of (1) is a first kind Volterra integral equation with solution $y \in C(I)$. It can be seen that the functions g and K_{21} of (1) are satisfy to the hypothesis of the theorem

$$g \in C^1(I), \quad g(0) = 0, \quad K_{21} \in C^1(D), \quad |\det(K_{21}(t,t))| \geq k0 > 0, \quad t \in I,$$

hence there exists an $h > 0$, such that for all meshes I_h, with diameter $h \in (0, \overline{h})$, the corresponding linear algebraic system arises from the spline collocation equation, possesses a unique solution and the collocation solution $\mathbf{u}$ converges to the exact solution $\mathbf{y}$, with the error estimation:

$$\|\mathbf{y} - \mathbf{u}\|_\infty = \begin{cases} O(h^m), & \text{if } \rho_m \in [-1, 1), \\ O(h^{m-1}), & \text{if } \rho_m = 1. \end{cases}$$

Actually, the convergence properties of $\mathbf{y}$ to $\mathbf{u}$, in this case may be discussed separately from $\mathbf{z}$. So, our claim is obtaining the attainable order of convergence of $\|\mathbf{z} - \mathbf{v}\|_\infty$.

Rewriting the Eq. (16), after inserting (17) and (10) get,

$$\begin{aligned}
\mathbf{0} = {} & \sum_{k=1}^{m} \int_0^{c_j} \mathbf{K}_{21}(t_{nj}, t_n + \rho h)\mathbf{E}_{nk} L_k(\rho) d\rho \\
& + \sum_{k=1}^{m} \int_{c_m}^{1} \mathbf{K}_{21}(t_{(n-1)m}, t_{n-1} + \rho h)\mathbf{E}_{(n-1)k} L_k(\rho) d\rho \\
& + \sum_{i=0}^{n-1} \sum_{k=1}^{m} \int_0^1 (c_j h + (1 - c_m)h) \frac{\partial \mathbf{K}_{21}}{\partial t}(t_n, t_i + \rho h)\mathbf{E}_{ik} L_k(\rho) d\rho \\
& + h^m \int_0^{c_j} \mathbf{K}_{21}(t_{nj}, t_n + \rho h)\mathbf{R}_{m,n}(\rho) d\rho \\
& + h^m \int_{c_m}^{1} \mathbf{K}_{21}(t_{(n-1)m}, t_{n-1} + \rho h)\mathbf{R}_{m,n}(\rho) d\rho \\
& + h^m \sum_{i=0}^{n-1} \int_0^1 (c_j h + (1 - c_m)h) \frac{\partial \mathbf{K}_{21}}{\partial t}(t_n, t_i + \rho h)\mathbf{R}_{m,n}(\rho) d\rho. \qquad (27)
\end{aligned}$$

The equations (20) and (27) can be rewritten in the matrix notation

$$\mathbf{A_1}\overline{\mathbf{E}}_n = \mathbf{Q}\overline{\mathbf{E}}_{n-1} + \sum_{i=0}^{n-1} \mathbf{G}'\overline{\mathbf{E}}_i + \mathbf{C}', \qquad (28)$$

where

$$\mathbf{Q} = \begin{pmatrix} \{0\}_{d_1 \times d_1} & \{0\}_{d_1 \times d_2} \\ \mathbf{Q_0} & \{0\}_{d_2 \times d_2} \end{pmatrix}, \quad \mathbf{Q_0} = \begin{pmatrix} -\displaystyle\int_{c_m}^{1} \mathbf{K}_{21}(t_{(n-1)m}, t_{n-1} + \rho h) L_k(\rho) d\rho \\ k = 1, \ldots, m \end{pmatrix},$$

$$\mathbf{G}' = \begin{pmatrix} h\mathbf{B}_{n,i}^{(1,1)} & h\mathbf{B}_{n,i}^{(1,2)} \\ -(c_j + (1 - c_m))h\tilde{\mathbf{B}}_{n,i}^{(2,1)} & \{0\}_{d_2 \times d_2} \end{pmatrix},$$

$$\mathbf{C}' = \begin{pmatrix} h^{m+1}\left(\mathbf{P}_n^{(1,1)} + \mathbf{P}_n'^{(1,2)} + \sum_{i=0}^{n-1}\mathbf{P}_{n,i}^{(1,1)} + \sum_{i=0}^{n-1}\mathbf{P}_{n,i}'^{(1,2)}\right) \\ h^m(\mathbf{Q_1} - \mathbf{P}_n^{(2,1)}) - c_j h^{m+1}\sum_{i=0}^{n-1}\tilde{\mathbf{P}}_{n,i}^{(2,1)} \end{pmatrix},$$

and $\mathbf{Q_1} = \int_{c_m}^1 \mathbf{K}_{21}(t_{(n-1)m}, t_{n-1} + \rho h)\mathbf{R}_{m,n}(\rho)d\rho$.

Multiplying (28) by $\mathbf{A_1^{-1}}$, the second equation of the resulting system is obtained as follows

$$\mathbf{E}_n' = \mathbf{D_2}\mathbf{E}_{n-1}' + \sum_{i=0}^{n-2}\mathbf{D_2}\mathbf{E}_i' + \mathbf{K}\mathbf{Q_0}\mathbf{E}_{n-1} + \sum_{i=0}^{n-1}\mathbf{D_1'}\mathbf{E}_i + \mathbf{O}(h^{m-1}), \tag{29}$$

where $\mathbf{K} = (F_4 - F_3F_1^{-1}F_2)^{-1}$.

Consequently, $\mathbf{B}_{n,i}^{(1,2)}$ may be written as

$$\mathbf{B}_{n,i}^{(1,2)} = \begin{pmatrix} \int_0^{c_m}\mathbf{K}_{12}(t_{nj}, t_i + \rho h)L_k(\rho)d\rho + \int_{c_m}^1 \mathbf{K}_{12}(t_{nj}, t_i + \rho h)L_k(\rho)d\rho \\ k = 1, \dots, m \end{pmatrix}.$$

From this, we can write $\mathbf{D_2} = \mathbf{D_3} + \mathbf{D_4}$, such that

$$\mathbf{D_3} = -h(F_4 - F_3F_1^{-1}F_2)^{-1}F_3F_1^{-1}\begin{pmatrix} \int_0^{c_m}\mathbf{K}_{12}(t_{nj}, t_i + \rho h)L_k(\rho)d\rho \\ k = 1, \dots, m \end{pmatrix}, \tag{30}$$

$$\mathbf{D_4} = -h(F_4 - F_3F_1^{-1}F_2)^{-1}F_3F_1^{-1}\begin{pmatrix} \int_{c_m}^1\mathbf{K}_{12}(t_{nj}, t_i + \rho h)L_k(\rho)d\rho) \\ k = 1, \dots, m \end{pmatrix}. \tag{31}$$

Using (30) and (31), the Eq. (29) can be written as

$$\mathbf{E}_n' = (\mathbf{D_4} + \mathbf{D_3})\mathbf{E}_{n-1}' + \sum_{i=0}^{n-2}\mathbf{D_2}\mathbf{E}_i' + \mathbf{K}\mathbf{Q_0}\mathbf{E}_{n-1} + \sum_{i=0}^{n-1}\mathbf{D_1'}\mathbf{E}_i + \mathbf{O}(h^{m-1}). \tag{32}$$

Now, we first rewrite $\mathbf{B}_n^{(1,2)}$ as

$$\mathbf{B}_n^{(1,2)} = \int_0^{c_j}\mathbf{K}_{12}(t_{nj}, t_n + \rho h)L_k(\rho)d\rho = \mathbf{K}_{12}(t_n, t_n)\bigotimes A + \{O(h)\}_{d_1 \times d_2},$$

where $A = \begin{pmatrix} \int_0^{c_j} L_k(\rho)d\rho \\ k, j = 1, \dots, m \end{pmatrix}$ and $\bigotimes$ is Kronecker product. Also, we have

$$\int_{c_m}^1 \mathbf{K}_{12}(t_{nj}, t_i + \rho h)L_k(\rho)d\rho = \mathbf{K}_{12}(t_i, t_i)\bigotimes F + \{O(h)\}_{d_1 \times d_2},$$

where $F = \begin{pmatrix} \int_{c_m}^{1} L_k(\rho)d\rho \\ k = 1,\ldots,m \end{pmatrix}$. On the other hand

$$\mathbf{D}_4 = -\big(F_3F_1^{-1}(\mathbf{K}_{12}(t_n,t_n) \bigotimes A + \{O(h)\}_{d_1\times d_2})\big)^{-1} F_3F_1^{-1}$$
$$\times \big(\mathbf{K}_{12}(t_i,t_i) \bigotimes F + \{O(h)\}_{d_1\times d_2}\big).$$

In order to describe the key ideas without having to resort to complex notation for $\mathbf{D}_4$, using properties of the Kronecker product we obtain

$$\mathbf{D}_4 \simeq \mathbf{D}_5 \bigotimes A^{-1}F,$$

where $\mathbf{D}_5$ is an identity $d_2 \times d_2$ matrix. According to Lemma 2.4.3 of Brunner [2004], we know that $A^{-1}F$ has a nontrivial eigenvalue as

$$\lambda = \rho_m = (-1)^m \prod_{i=1}^{m} \frac{1-c_i}{c_i}.$$

It follows from the elementary theory of the difference equations [see Elaydi (1999)] that the solutions of the system of the first-order difference Eq. (32) remain uniformly bounded, if and only if $|\lambda| = |\rho_m| \le 1$. [For further details see Brunner (2004), pp. 126]

Now, if $\rho_m = [-1,1)$ then by considering Theorem 2.4.2 from Brunner [2004], we conclude $\|\mathbf{E}_n\|_1 = O(h^m), (n = 0,\ldots,N-1)$ so, the Eq. (32) can be written as:

$$\mathbf{E}'_n = (\mathbf{D_4} + \mathbf{D_3})\mathbf{E}'_{n-1} + \sum_{i=0}^{n-2} \mathbf{D_2}\mathbf{E}'_i + \mathbf{O}(h^{m-1}). \tag{33}$$

Since $A^{-1}F$ is diagonalizable, there exists a nonsingular matrix P such that $\mathbf{D_4} \simeq \mathbf{D}_5 \bigotimes PD_\lambda P^{-1}$ where $D_\lambda = \mathrm{diag}(\lambda, 0, \ldots, 0)$. Multiplying (33) by $\widehat{\mathbf{P}}_{d_2\times d_2} = \mathrm{diag}(P^{-1}, P^{-1}, \ldots, P^{-1})$ and defining $\mathbf{Z}_n = \widehat{\mathbf{P}}\mathbf{E}'_n$, we obtain

$$\mathbf{Z}_n = \big(\mathbf{D}_5 \bigotimes D_\lambda\big)\mathbf{Z}_{n-1} + \widehat{\mathbf{P}}\mathbf{D_3}\widehat{\mathbf{P}}^{-1}\mathbf{Z}_{n-1} + \sum_{i=0}^{n-2} \widehat{\mathbf{P}}\mathbf{D_2}\widehat{\mathbf{P}}^{-1}\mathbf{Z}_i + \mathbf{O}(h^{m-1}). \tag{34}$$

Taking norms, we arrive at inequality of the form

$$\|\mathbf{Z}_n\| \le |\lambda|\|\mathbf{Z}_{n-1}\| + \gamma_3\|\mathbf{Z}_{n-1}\| + \gamma_2 \sum_{i=0}^{n-2} \|\mathbf{Z}_i\| + O(h^{m-1}). \tag{35}$$

Repeated insertion of this formula and considering the Lemma 6 of Hairer *et al.* [1983] give

$$\|\mathbf{Z}_n\| \le |\lambda|^n\|\mathbf{Z}_0\| + \sum_{i=1}^{n} |\lambda|^{n-i} O(h^{m-1}) + O(h^{m-1}). \tag{36}$$

If $|\lambda| < 1$, then $\|\mathbf{Z}_n\| = O(h^{m-1})$, and

$$\|\mathbf{E}'_n\|_1 = O(h^{m-1}). \tag{37}$$

If $|\lambda| = 1$, then $\|\mathbf{Z}_n\| = O(h^{m-2})$, and

$$\|\mathbf{E}'_n\|_1 = O(h^{m-2}). \tag{38}$$

In the case of $\rho_m = 1$, by considering Theorem 2.4.2 from Brunner [2004], we have $\|\mathbf{E}_n\| = O(h^{m-1}), (n = 0, \ldots, N-1)$ and the Eq. (32) can be written as

$$\mathbf{E}'_n = (\mathbf{D_4} + \mathbf{D_3})\mathbf{E}'_{n-1} + \sum_{i=0}^{n-2} \mathbf{D_2}\mathbf{E}'_i + \mathbf{O}(h^{m-2}). \tag{39}$$

Using a similar procedure as outlined for (34), (35) and (36), the following result can be obtained for the relation (39)

$$\|\mathbf{E}'_n\|_1 = O(h^{m-3}). \tag{40}$$

Now, (37) and (26) for $c_m < 1$ and $\rho_m = (-1, 1)$, give

$$\|\mathbf{z} - \mathbf{v}\|_\infty = O(h^{m-1}).$$

Also, (38) and (26) for $c_m < 1$ and $\rho_m = -1$ (m is odd), yield

$$\|\mathbf{z} - \mathbf{v}\|_\infty = O(h^{m-2}).$$

Finally, using (40) and (26) for $c_m < 1$, and $\rho_m = 1$ (m is even), we conclude

$$\|\mathbf{z} - \mathbf{v}\|_\infty = O(h^{m-3}),$$

which leads to the stated estimates of the theorem. □

We must emphasize that, the convergence orders in Theorem 2 are completely different from the analogous one for index-1 IAEs in Kauthen [2000]. It is interesting to note that for the index-2 IAEs (1) in the case of $c_m < 1$, the orders of convergence $O(h^{m-2})$ and $O(h^{m-3})$ are attained for z-component, if m is odd ($\rho_m = -1$) and even ($\rho_m = 1$), respectively. However, following kauthen [2000], for IAEs of index-1 the global convergence orders $O(h^m)$ and $O(h^{m-1})$ for z have been obtained while $-1 \le \rho_m < 1$ and $\rho_m = 1$, respectively.

4. Numerical Results and Discussions

To illustrate the theoretical findings of the preceding sections, we will present some numerical results of the proposed method for the considered IAEs systems of index-2 from Hadizadeh *et al.* [2011]. Let $\mathbf{u} = (u_1, \ldots, u_{d_1})^T$ and $\mathbf{v} = (v_1, \ldots, v_{d_2})^T$ with $\{u_l\}_{l=1}^{d_1}, \{v_l\}_{l=1}^{d_2} \in S_{m-1}^{-1}(\Pi_N)$ be the collocation solution. We consider the collocation parameters in two cases, $c_m = 1$ and $c_m < 1$. For $c_m = 1$, we choose the Radau II points (i.e., the zeros of $P_{m-1}(2s-1) - P_m(2s-1)$) as collocation parameters and for the case $c_m < 1$, we apply the zeros of $P_m(2s-1)$ as the collocation parameters in which P_m denotes the Legendre polynomial of degree m. The accuracy of the proposed method is measured by computing the difference between the exact and the numerical solution at each gridpoints. We also report the observed order of convergence from the maximum errors $\|\mathbf{y}-\mathbf{u}\|_\infty$ and $\|\mathbf{z}-\mathbf{v}\|_\infty$ at the gridpoints. All the calculations were supported by the Mathematica®.

Example 1. [from Hadizadeh *et al.* (2011)] Consider the following system of linear semi-explicit index-2 IAEs of the Hessenberg form:

$$AX(t) = G(t) + \int_0^t K(t,s)X(s)ds, \quad t \in [0,1],$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad K(t,s) = \begin{pmatrix} e^{t+s} & (s+1)^2 \\ s+t+2 & 0 \end{pmatrix},$$
$$X(t) = (y(t),\ z(t))^T, \quad G(t) = (f(t),\ g(t))^T,$$

and

$$\begin{aligned} f(t) &= \sin t - \frac{1}{2}e^t(1 + e^t(-\cos t + \sin t)) \\ &\quad - \frac{1}{4}(-2 + 2(1+t)\cos 2t + (1 + 4t + 2t^2)\sin 2t), \\ g(t) &= -(2+t) + 2(1+t)\cos t - \sin t. \end{aligned}$$

The exact solution of the system is:

$$y(t) = \sin t, \quad z(t) = \cos 2t.$$

The maximum errors have been shown for different values of m and N at the gridpoints in the Tables 1, 2, 5, and 6. Also, the orders of convergence from the maximum errors at the gridpoints have been reported in Tables 3, 4, 7, and 8 which they confirm the theoretical results of the Theorem 2.

Example 2. [from Hadizadeh *et al.* (2011)]

$$AX(t) = G(t) + \int_0^t K(t,s)X(s)ds, \quad t \in [0,1],$$

Table 1. Maximum errors $\|y-u\|_\infty$ and $\|z-v\|_\infty$ with $c_m = 1$ in Example 1.

	$\|y-u\|_\infty$			$\|z-v\|_\infty$		
m	$N = 16$	$N = 32$	$N = 64$	$N = 16$	$N = 32$	$N = 64$
2	4.00E − 4	1.04E − 4	2.65E − 5	2.69E − 2	1.68E − 2	9.38E − 3
3	4.11E − 6	5.16E − 7	6.46E − 8	4.63E − 4	1.13E − 4	2.78E − 5
4	2.73E − 8	1.76E − 9	1.11E − 10	6.58E − 6	9.82E − 7	1.33E − 7

Table 2. Maximum errors $\|y-u\|_\infty$ and $\|z-v\|_\infty$ with $c_m<1$ in Example 1.

	$\|y-u\|_\infty$			$\|z-v\|_\infty$		
m	$N=16$	$N=32$	$N=64$	$N=16$	$N=32$	$N=64$
3	6.75E − 6	8.38E − 7	1.04E − 7	4.10E − 3	2.00E − 3	9.95E − 4
4	3.69E − 7	4.65E − 8	5.83E − 9	7.24E − 4	3.65E − 4	1.82E − 4

Table 3. Orders of convergence of u and v with $c_m=1$ in Example 1.

	u			v		
m	$N=16$	$N=32$	$N=64$	$N=16$	$N=32$	$N=64$
2	1.87	1.96	1.97	0.36	0.67	0.84
3	2.98	2.99	2.99	2.05	2.03	2.02
4	3.90	3.95	3.98	2.79	2.79	2.88

Table 4. Orders of convergence of u and v with $c_m<1$, $\rho_m=-1$ and $\rho_m=1$ in Example 1.

	u			v		
m	$N=16$	$N=32$	$N=64$	$N=16$	$N=32$	$N=64$
3	3.02	3.00	3.00	1.10	1.03	1.00
4	2.97	2.98	2.99	0.94	0.98	1.00

Table 5. Maximum errors $\|y_1-u_1\|_\infty, \|y_2-u_2\|_\infty$ and $\|z_1-v_1\|_\infty$ with $c_m=1$ in Example 2.

	$\|y_1-u_1\|_\infty$			$\|y_2-u_2\|_\infty$			$\|z_1-v_1\|_\infty$		
m	$N=16$	$N=32$	$N=64$	$N=16$	$N=32$	$N=64$	$N=16$	$N=32$	$N=64$
2	6.08E − 3	1.54E − 3	3.89E − 4	3.08E − 3	7.79E − 4	1.95E − 4	2.79E − 1	1.45E − 1	7.42E − 2
3	7.93E − 5	1.01E − 5	1.28E − 6	3.96E − 5	5.07E − 6	6.40E − 7	3.82E − 3	9.75E − 4	2.46E − 4
4	6.06E − 7	3.86E − 8	2.44E − 9	3.03E − 7	1.93E − 8	1.22E − 9	9.70E − 5	1.23E − 5	1.56E − 6

Table 6. Maximum errors $\|y_1-u_1\|_\infty, \|y_2-u_2\|_\infty$ and $\|z_1-v_1\|_\infty$ with $c_m<1$ in Example 2.

	$\|y_1-u_1\|_\infty$			$\|y_2-u_2\|_\infty$			$\|z_1-v_1\|_\infty$		
m	$N=16$	$N=32$	$N=64$	$N=16$	$N=32$	$N=64$	$N=16$	$N=32$	$N=64$
3	7.93E − 5	1.04E − 5	1.33E − 6	4.89E − 5	6.76E − 6	8.89E − 7	8.15E − 2	4.08E − 2	2.04E − 2
4	1.93E − 5	2.47E − 6	3.13E − 7	1.27E − 5	1.65E − 6	2.10E − 7	3.47E − 2	1.78E − 2	9.06E − 3

Table 7. Order of convergence of $u_1,\ u_2$, and v_1 with $c_m = 1$ in Example 2.

	u_1			u_2			v_1		
m	$N = 16$	$N = 32$	$N = 64$	$N = 16$	$N = 32$	$N = 64$	$N = 16$	$N = 32$	$N = 64$
2	1.94	1.98	1.98	1.96	1.98	1.99	0.88	0.94	0.96
3	2.93	2.97	2.98	2.93	2.96	2.98	1.94	1.97	1.98
4	3.94	3.97	3.98	3.93	3.97	3.98	2.93	2.97	2.98

Table 8. Orders of convergence of $u_1,\ u_2$, and v_1 with $c_m < 1$, $\rho_m = -1$ and $\rho_m = 1$ in Example 2.

	u_1			u_2			v_1		
m	$N = 16$	$N = 32$	$N = 64$	$N = 16$	$N = 32$	$N = 64$	$N = 16$	$N = 32$	$N = 64$
3	2.86	2.93	2.96	2.73	2.85	2.92	0.99	0.99	1
4	2.93	2.96	2.98	2.90	2.94	2.97	0.92	0.96	0.97

where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K(t,s) = \begin{pmatrix} \dfrac{3-2s}{2-s} & \dfrac{3-s}{2-s} & 2(2-s) \\ \dfrac{-1}{s-2} & -1 & 1 \\ s+2 & s^2-4 & 0 \end{pmatrix},$$

$$X(t) = (y_1(t), y_2(t), z_1(t))^T, \quad G(t) = (1, 2e^t - 1, -1 + e^t(-t^2 + t + 1))^T,$$

with the exact solution:

$$y_1(t) = y_2(t) = e^t, \quad z_1(t) = -\frac{e^t}{2-t}.$$

We compare the results obtained by the presented method with those obtained by Jacobi spectral method in Hadizadeh *et al.* [2011]. However, we note that, spectral methods are global methods such that the computation at any given point depends not only on information at neighboring points, but on information from the entire domain. Due to completely smoothness of the exact solutions for two previous examples, the results obtained using Jacobi spectral method Hadizadeh *et al.* [2011] is more accurate than the spline collocation method in this case. The error behaviors obtained by the spline collocation and Jacobi spectral methods for the special values of m and N in Examples 1, 2 are shown in Figs. 1 and 2, respectively.

In this position, for the better comparison purpose, we present the following test problem:

Example 3. Consider the following IAEs system:

$$AX(t) = G(t) + \int_0^t K(t,s)X(s)ds, \quad t \in [0,1],$$

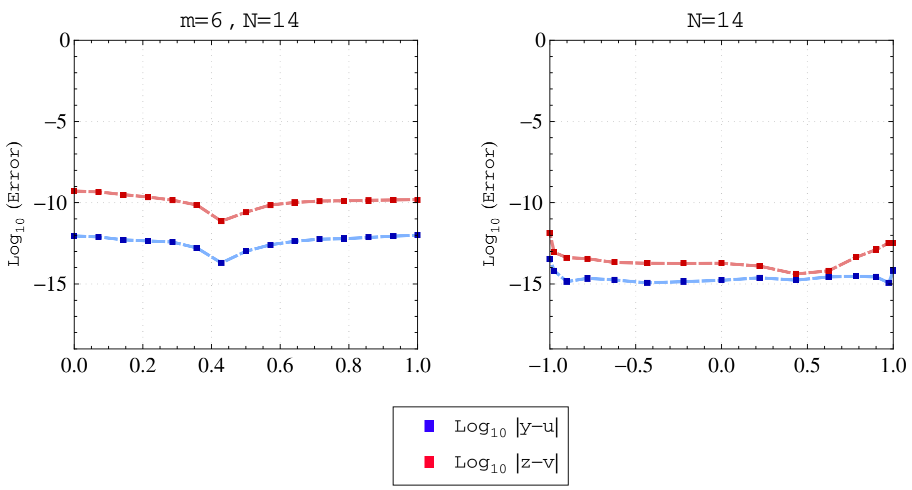

Fig. 1. On the left we show the numerical results of the spline collocation method with $N = 14$, $c_m = 1$, and $m = 7$ in Example 1. On the right we show the numerical results of the same problem using Jacobi spectral method for $N = 14$.

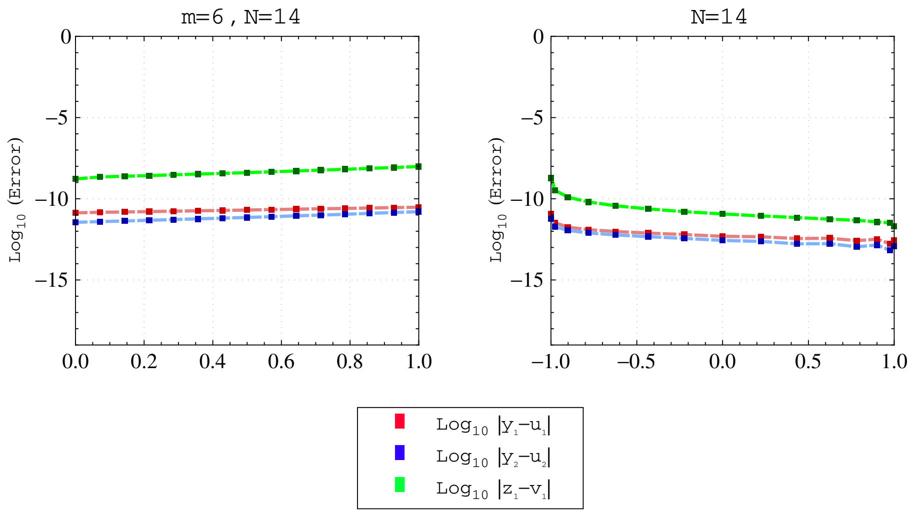

Fig. 2. On the left we show the numerical results of the spline collocation method with $N = 14$, $c_m = 1$ and $m = 7$ in Example 2. On the right we show the numerical results of the same problem using Jacobi spectral method for $N = 14$.

where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad K(t,s) = \begin{pmatrix} t + s^2 & (s + t^2 + 1) \\ s^2 + t^4 + 4 & 0 \end{pmatrix},$$

$$X(t) = (y(t), z(t))^T, \quad G(t) = (f(t), g(t))^T,$$

and f, g such that the exact solution is

$$y(t) = \frac{t}{t^4+1}, \quad z(t) = \arctan(t^2+1).$$

The problem is solved by the Jacobi collocation method Hadizadeh *et al.* [2011], with $\alpha = \frac{1}{2}, \beta = \frac{1}{3}$ and then the $L^2_{w^{\alpha,\beta}}$ errors for different values of $N = 6, 8, \ldots$ are reported in Table 9. Again, it is solved by the spline collocation method and the maximum errors and order of convergence for different values of m and N at the gridpoints are shown in Tables 10–13. Figure 3 shows a comparison between the spline collocation scheme with $m = 6,\ N = 14$ and the Jacobi spectral method for

Table 9. $L^2_{w^{\alpha,\beta}}$ errors for Example 3.

N	$\|y-u\|_{L^2_{w^{\alpha,\beta}}}$	$\|z-v\|_{L^2_{w^{\alpha,\beta}}}$
6	6.41E − 4	7.02E − 3
8	4.84E − 5	6.60E − 4
10	1.97E − 6	2.59E − 5
12	2.13E − 7	1.24E − 5
14	5.70E − 8	3.03E − 6

Table 10. Maximum errors $\|y-u\|_\infty$ and $\|z-v\|_\infty$ with $c_m = 1$ in Example 3.

	$\|y-u\|_\infty$			$\|z-v\|_\infty$		
m	$N = 16$	$N = 32$	$N = 64$	$N = 16$	$N = 32$	$N = 64$
2	1.18E − 3	2.97E − 4	7.46E − 5	4.90E − 2	2.53E − 2	1.28E − 2
3	2.65E − 5	3.41E − 6	4.28E − 7	1.55E − 3	3.78E − 4	9.45E − 5
4	8.70E − 7	5.48E − 8	3.44E − 9	1.18E − 4	1.52E − 5	1.94E − 6

Table 11. Maximum errors $\|y-u\|_\infty$ and $\|z-v\|_\infty$ with $c_m < 1$ in Example 3.

	$\|y-u\|_\infty$			$\|z-v\|_\infty$		
m	$N = 16$	$N = 32$	$N = 64$	$N = 16$	$N = 32$	$N = 64$
3	3.66E − 5	4.48E − 6	5.56E − 7	1.30E − 2	6.18E − 2	3.03E − 3
4	5.99E − 6	7.49E − 7	9.40E − 8	1.21E − 2	6.07E − 3	3.04E − 3

Table 12. Orders of convergence of u and v with $c_m = 1$ in Example 3.

	u			v		
m	$N = 16$	$N = 32$	$N = 64$	$N = 16$	$N = 32$	$N = 64$
2	1.97	1.99	1.99	0.87	0.95	0.98
3	2.90	2.95	2.99	2.08	2.03	2.00
4	3.91	3.98	3.99	2.75	2.95	2.96

Table 13. Orders of convergence of u and v with $c_m < 1$, $\rho_m = -1$ and $\rho_m = 1$ in Example 3.

	u			v		
m	$N = 16$	$N = 32$	$N = 64$	$N = 16$	$N = 32$	$N = 64$
3	3.08	3.03	3.01	1.22	1.07	1.02
4	3.01	2.99	2.99	1.00	0.99	1.00

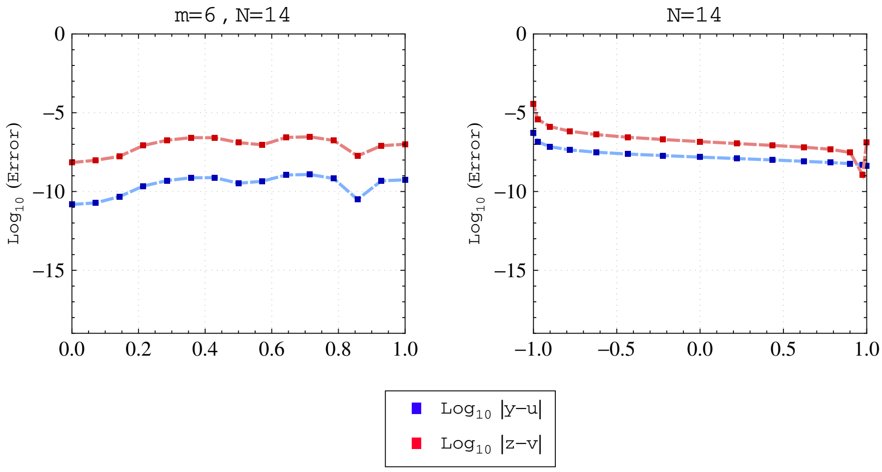

Fig. 3. On the left we show the numerical results of the spline collocation method with $N = 14$, $c_m = 1$ and $m = 7$ in Example 3. On the right we show the numerical results of the same problem using Jacobi spectral method for $N = 14$.

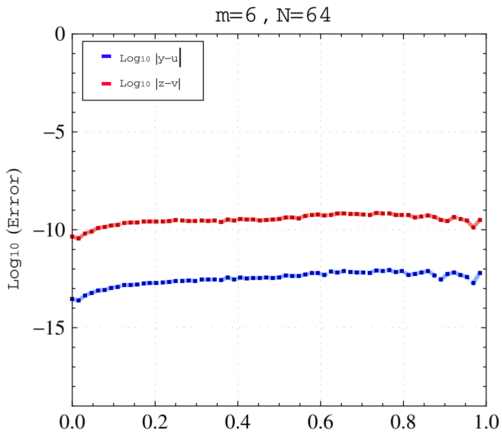

Fig. 4. Numerical results of the spline collocation method with $N = 64$, $c_m = 1$, and $m = 6$ in Example 3.

$N = 14$. As it can be seen from Fig. 3, the spline collocation method is superior in accuracy to the Jacobi spectral method in this case.

Noting that, increasing m in the spline collocation method improve the accuracy, e.g., for $m = 6$ and $N = 64$, we have

$$\|y - u\|_\infty = 8.63E - 13, \quad \|z - v\|_\infty = 7.26E - 10.$$

Figure 4, represents the error behavior for these values of m and N.

5. Conclusion

In this paper, we performed a polynomial spline collocation method for the numerical solution of system of semi-explicit IAEs of index-2. The most important novelty of this work is obtaining the necessary and sufficient conditions for convergence of discontinuous collocation approximate solution for the IAEs system. We emphasize that in the case of IAEs of index-1, it has shown that the superconvergence occurs when the collocation parameters c_j are the Radau II points for (0,1] [see Kauthen (2000) for further details], however our numerical experiments show that for the system of IAEs of index-2, this property does not occur in these points. More investigation of the superconvergence results for the IAEs of index-2 will be the subject of our future work.

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