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New enclosure algorithms for the verified solutions of nonlinear Volterra integral equations

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ABSTRACT

This paper is concerned with new algorithms which provide the sharp bounds that are guaranteed to contain the exact solutions of nonlinear Volterra integral equations. We develop new enclosure algorithms based on the interval methods which was first introduced by Moore in [24] together with the Taylor polynomials to improve the accuracy of the scheme by reducing the width of interval solutions. The modified methods calculate a priori bound automatically in parallel with the computation of solutions of integral equations. We will show that the accuracy of the proposed algorithms is dependent on the number of interval subdivisions. Some numerical experiments are also included to demonstrate the validity and applicability of the scheme and showing a marked improvement in comparison with the recent existing numerical results.

1. Introduction

In recent years, enclosure methods have been used widely in numerical solution of many operator equations (see e.g. [1–5]). These methods usually start with an interval vector which contains a solution and improve this inclusion iteratively. In comparison to classical numerical schemes which only provide the approximate values of the solutions and they rarely determine the precise error bounds, interval enclosure methods enable us to compute guaranteed error bounds including all discretization and round off errors. It is known that truncation and round off errors corresponding to the problem are usually negligible but they may cause some serious problems (see e.g. [6]). This makes computation with guaranteed error bounds a useful tool in applied problems in science and technology. Today many generalized models of uncertainty are closely connected with interval analysis e.g. fuzzy set theory and engineering problems with interval valued parameters or interval initial values [4], so interval algorithms are powerful tools for analyzing such problems. Some applications of interval methods in robust control, robotics, multimedia architectures and other engineering problems may be found in [7–9]. (see also [1] for a survey on enclosure methods).

On the other hand, nonlinear integral equations usually originate from mathematical descriptions of natural phenomena such as elastic systems, optical systems and the theory of biological and population growth [10–12]. Here, we focus on computation of the guaranteed bounds for the solutions of the following nonlinear Volterra integral equation

$$u(x) = f(x) + \int_0^x k(x,t,u(t))dt, \quad x \in [0,\mathcal{X}], \tag{1.1}$$

where f and k are given continuous functions on $[0,\mathcal{X}]$ and $S \times \mathbb{R}$ (where $S = \{(x,t) : 0 \leqslant t \leqslant x \leqslant \mathcal{X}\}$, respectively and $u(x)$ is a solution to be determined.

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Some commonly used methods that are often used to determine numerical approximations to (1.1) may be found in [13–15]. There are also some considerable interval based works in solving Fredholm and Volterra integral equations. Moore has introduced an iterative algorithm for nonlinear integral equations based on the definition of interval integration, however this method is not so efficient and has a slow rate of convergence [24, pp. 88]. Caprani et al. in [16] presented a similar method to solve the general form of Fredholm integral equations by defining a nested sequence of function intervals with the contraction property of the integral operator using the mean value form of the operator. Dobner published several papers on solving Fredholm integral equations using enclosure methods [17,18]. In 2003, Berz and Makino [19] employed a new version of the method called *Taylor model*. Recently, Murashige and Oishi in [20,21] presented numerical verification of solutions of periodic integral equations with a singular kernel and Nekrasovs integral equation. In comparison to Fredholm type, there are only few methods for solving Volterra integral equations, especially when they are nonlinear [22,23].

In the present work, we provide high order guaranteed bounds for the solutions of nonlinear Volterra integral equations of the second kind. Our aim is applying the Taylor polynomials with bounded remainder to enclose the integral kernel and other functions (instead of integral operator) and also computing the enclosures for partial derivatives based on initial intervals. We develop new enclosure algorithms based on Moore's interval method [24] together with the Taylor polynomials to increase the accuracy of the scheme and reducing the width of interval solutions.

It is worth noting that the algorithms of this paper can be employed to solve any nonlinear problems (whether the nonlinearity be algebraic, exponential and etc.) that may arise in practice.

The organization of this paper is as follows. We first summarize the principles of interval arithmetic and some relevant topics. Some modified enclosure algorithms including the Taylor polynomials to improve the accuracy of the scheme and the convergence conditions of the iterations are given in Section 3. Finally, in the last section some numerical experiments are reported to clarify the method and a few comparisons are made with existing classical methods in the literature.

2. Preliminaries

We first introduce some basic properties of interval arithmetic from [17,3–5]. An interval number is a closed set in $\mathbb{R}$ that includes the possible range of an unknown real number, where $\mathbb{R}$ denotes the set of real numbers. Therefore, a real interval is a set of the form

$$X = [\underline{x}, \bar{x}],$$

where $\underline{x}$ and $\bar{x}$ are the lower and upper bounds (endpoints) of the interval number X, respectively. The set of compact real intervals is denoted by

$$\mathbb{IR} = \{X = [\underline{x}, \bar{x}] | \underline{x},\ \bar{x} \in \mathbb{R},\ \underline{x} \leqslant \bar{x}\}.$$

A real number x is identified with a point interval $X = [x,x]$. The quality of interval analysis is measured by the width of the interval results, and a sharp enclosure for the exact solution is desirable. The *midpoint* and the *width* of an interval X are denoted by $m(X) := (\underline{x} + \bar{x})/2$ and $w(X) := \bar{x} - \underline{x}$, respectively. Considering $|X| = \max\{|\underline{x}|, |\bar{x}|\}$, for any $X,\ Y \in \mathbb{IR}$ and $a,\ b \in \mathbb{R}$ we can conclude that

$$w(aX + bY) = |a|w(X) + |b|w(Y), \tag{2.1}$$

$$w(XY) \leqslant |X|w(Y) + |Y|w(X). \tag{2.2}$$

The four elementary operations of real arithmetic can be extended to intervals. Operations over intervals $\diamondsuit \in \{+, -, *, /\}$ are defined by the general rule

$$X \diamondsuit Y = \{x \diamondsuit y | x \in X,\ y \in Y\}.$$

It is easy to see that the set of all possible results when applying an operator $\diamondsuit$ to X and Y, forms a closed interval (for $0 \notin Y$ in the case of division) and the end point can be calculated by

$$X \diamondsuit Y = \{\min(x \diamondsuit y),\ \max(x \diamondsuit y)\}, \quad \textit{for}\ \diamondsuit \in \{+, -, *, /\}.$$

Now, we can define functions of interval variables or inclusion functions of f, in a similar fashion, as

$$F(X) = \{f(x) | x \in X\}. \tag{2.3}$$

Note that when F is an inclusion function of f, then we can directly obtain lower and upper bounds of F over any interval X within the domain of F just by taking $\underline{F}(X)$ and $\overline{F}(X)$, respectively.

Moore in [24] has introduced the interval extension of a real integral as

$$\int_a^x f(x)dx \in F([a,x])(x - a), \tag{2.4}$$

where X is the interval of integration and F is inclusion function of f.

In interval analysis the *dependency effect*, which is the failure of interval arithmetic to identify different occurrences of the same variable, usually results in overestimation in computations. For example, the range of $f(x) = x/(1 + x)$ on $X = [1,2]$ is [1/

2, 2/3], but $F(X) = [1/3, 1]$. To overcome this difficulty, some alternative schemes such as centered form and Taylor polynomials are proposed in [1,24,5]. Also during the last 10 years, significant progress has been made in bounding function ranges and control of the dependency problem of interval arithmetic. This is known as the Taylor model. In this method, a function is represented using a model consisting of a Taylor polynomial together with an interval remainder bound. For further details see e.g. [19,25].

Here, we will use the terminology of interval polynomial enclosures and truncated Taylor series together with bounded remainders to reduce the overestimation.

2.1. Bounding the remainder in Taylor polynomials

Let f have continuous derivatives of any necessary order. It is known that by expanding $f(y)$ about a point x we may write

$$f(y) = f(x) + (y-x)f'(x) + \cdots + \frac{(y-x)^m}{m!}f^{(m)}(x) + R_m(x,y,\xi),$$

where the remainder term in the Lagrange form is

$$R_m(x,y,\xi) = \frac{(y-x)^{m+1}}{(m+1)!}f^{(m+1)}(\xi), \quad \xi \in [x,y].$$

Hence, if $x, y \in X$ then $\xi \in X$, so $f^{(m+1)}(\xi) \in F^{(m+1)}(X)$ and

$$R_m(x,y,X) = \frac{(y-x)^{m+1}}{(m+1)!}F^{(m+1)}(X),$$

bounds the remainder for any $x, y \in X$. According to [26], we can conclude the following Taylor polynomial enclosures for $x_0 \in X$ and all $x \in X$ with $m = 0, 1$:

$$f(x) \in F_{T_1}(X) := f(x_0) + (X - x_0)F'(X), \tag{2.5}$$

$$f(x) \in F_{T_2}(X) := f(x_0) + (X - x_0)f'(x_0) + \frac{(X-x_0)^2}{2!}F''(X). \tag{2.6}$$

Also, similar relations hold for multidimensional cases. See [26] for more enclosures.

3. Method and background

Consider the nonlinear Volterra integral equation (1.1) where the following assumptions are satisfied:

(a) the function $k(x,t,u(t))$ is integrable and bounded in the domain of integration,
(b) the following Lipschitz condition is satisfied by $k(x,t,u(t))$ within its domain of definition

$$|k(x,t,u_1(t)) - k(x,t,u_2(t))| \leqslant L|u_1(t) - u_2(t)|. \tag{3.1}$$

Under these conditions Eq. (1.1) has a unique solution. Further details regarding existence and uniqueness results may be found in [27,12]. Throughout this paper, we assume that the above mentioned conditions are satisfied and the functions f and k are at least two times continuously differentiable on $[0, \mathcal{X}]$.

3.1. Zero order algorithm

Let Π_n be a uniform partition of the interval $[0, \mathcal{X}]$ with grid points

$$0 = x_0 < x_1 < \cdots < x_p = \mathcal{X},$$

we define the subintervals

$$X_l = [x_{l-1}, x_l], \quad l = 1, \ldots, p.$$

In order to provide guaranteed bounds for the solution $u(x)$ of (1.1), we present the following lemma:

Lemma 1. *Let $u(x) \in U_0 = [\underline{u_0}, \overline{u_0}]$ and for $x \in X_i$, $t \in X_l$, $1 \leqslant l \leqslant i \leqslant p$, assume that $f(x) \in F(X_i)$, $k(x,t,u(t)) \in K(X_i, X_l, U_0)$, then we can obtain a new enclosure for each $u(x)|_{x \in X_i}$ as follows*

$$u(x)|_{x\in X_i} \in U_{1,i} := F(X_i) + \sum_{l=1}^{i-1} K(X_i, X_l, U_0)w(X_l) + K(X_i, X_i, U_0)[0, w(X_i)].$$

Proof. From (1.1) and due to the properties of integration, for $x \in X_i$, we have

$$u(x) = f(x) + \int_0^x k(x,t,u(t))dt = f(x) + \sum_{l=1}^{i-1} \int_{x_{l-1}}^{x_l} k(x,t,u(t))dt + \int_{x_{i-1}}^{x} k(x,t,u(t))dt.$$

Using (2.4) and the lemma's hypothesis, we replace x by X_i to obtain an interval enclosure

$$u(x)|_{x \in X_i} \in U_{1,i} := F(X_i) + \sum_{l=1}^{i-1} K(X_i, X_l, U_0)(x_l - x_{l-1}) + K(X_i, X_l, U_0)(X_i - x_{i-1}), \tag{3.2}$$

Note that

$$X_i - x_{i-1} = [x_{i-1}, x_i] - x_{i-1} = [x_{i-1} - x_{i-1}, x_i - x_{i-1}] = [0, w(X_i)], \tag{3.3}$$

so, the relation (3.2), can be written as

$$U_{1,i} := F(X_i) + \sum_{l=1}^{i-1} K(X_i, X_l, U_0) w(X_l) + K(X_i, X_i, U_0)[0, w(X_i)], \tag{3.4}$$

and this proves the lemma. □

It should be noted that since the grid points $x_0, \ldots, x_p$ interior to $[0, \mathcal{X}]$ are common endpoints of two subinterval X_i and X_{i+1}, then

$$u(x_i) \in U_{1,i} \cap U_{1,i+1}.$$

Now, if we consider $U_{1,i} \cap U_0$ as a new approximation for $u(x)$, this can be substituted into the right hand side of (3.4) to give the next approximation. This process is then repeated to achieve a rigorous solution. Summarizing the above relations, we produce the following algorithm:

Algorithm 1. Zero order approximation

Input:
  Number of subdivisions p;
  Number of iterations n;
  U_0;
begin
  For $k = 0, \ldots, n$:
    For $i = 0, \ldots, p-1$:
      Compute $U_{1,i}$ from (3.4);
      $U_0 \leftarrow U_{1,i} \cap U_0$;
$\forall i$, $u(x_i) \in U_{1,i} \cap U_{1,i+1}$;
end.

This confirms the Moore's algorithm which is given in [24]. In order to investigate the convergence of the iterations, we make use of some of Moore's results regarding the conditions for convergence. Actually, as stated in [24, pp. 84] the algorithm is convergent in $[0, \mathcal{X}]$, if $\mathcal{X}L < 1$, where L is the Lipschitz constant introduced in (3.1).

3.2. The modified algorithm

Following the results of [24,28], the method described in previous section is not convergent in large domains and has a slow rate of convergence that does not yield sharp bounds for the solutions. The major cause of these difficulties is the *dependency problem* which makes an overestimation. Here, in order to increase the approximation order derived from the method and to reduce any overestimation, we will provide the modified algorithms consisting the interval polynomial enclosures and truncated Taylor series with bounded remainders. We first need to introduce some relevant definitions and pertinent results as follows:

$$F'(X_i) := \{f'(x) | x \in X_i\}, \quad F''(X_i) := \{f''(x) | x \in X_i\},$$

and for multidimensional case

$$K_x(X_i, X_l, U_0) := \left\{ \frac{d}{dx} k(x,t,u(t)) | x \in X_i, \ t \in X_l, \ u(t) \in U_0 \right\},$$

$$K_t(X_i, X_l, U_0) := \left\{ \frac{d}{dt} k(x,t,u(t)) | x \in X_i, \ t \in X_l, \ u(t) \in U_0 \right\},$$

$$K_{xx}(X_i, X_l, U_0) := \left\{ \frac{d^2}{dx^2} k(x, t, u(t)) | x \in X_i,\ t \in X_l,\ u(t) \in U_0 \right\},$$
$$K_{tt}(X_i, X_l, U_0) := \left\{ \frac{d^2}{dt^2} k(x, t, u(t)) | x \in X_i,\ t \in X_l,\ u(t) \in U_0 \right\},$$
$$K_{xt}(X_i, X_l, U_0) := \left\{ \frac{d^2}{dxdt} k(x, t, u(t)) | x \in X_i,\ t \in X_l,\ u(t) \in U_0 \right\}.$$

We can now state the following theorem which gives new interval enclosures:

Theorem 1. *Under the stated notations and assumptions of the* Lemma 1, *we can obtain a new enclosure for each* $u(x)|_{x \in X_i}$ *as*

$$u(x)|_{x \in X_i} \in U^{\gamma}_{1,i} := F^{\gamma}_i + \sum_{l=1}^{i-1} K^{\gamma}_{ij} w(X_l) + K^{\gamma}_{ii}[0, w(X_i)], \quad \gamma = 1, 2, \tag{3.5}$$

where

$$F^1_i := f(x_{i-1}) + [0, w(X_i)] F'(X_i),$$
$$K^1_{ij} := k(x_{i-1}, t_{l-1}, \underline{u_0}) + [0, w(X_i)] K_x(X_i, X_l, U_0) + [0, w(X_l)] K_t(X_i, X_l, U_0),$$

and

$$F^2_i := f(x_{i-1}) + [0, w(X_i)] f'(x_{i-1}) + \frac{1}{2}[0, w(X_i)]^2 F''(X_i),$$

$$K^2_{ij} := k(x_{i-1}, t_{l-1}, \underline{u_0}) + [0, w(X_i)] \frac{d}{dx} k(x_{i-1}, t_{l-1}, \underline{u_0}) + [0, w(X_l)] \frac{d}{dt} k(x_{i-1}, t_{l-1}, \underline{u_0}) + \frac{1}{2}[0, w(X_i)]^2 K_{xx}(X_i, X_l, U_0) + \frac{1}{2}[0, w(X_l)]^2 K_{tt}(X_i, X_l, U_0) + [0, w(X_i) w(X_l)] K_{xt}(X_i, X_l, U_0).$$

Proof. We first apply Taylor polynomial enclosures to $f(x)$ and $k(x,t,u(x))$ which were introduced in (2.5) and (2.6):

$$f(x)|_{x \in X_i} \in F^1_i := f(x_{i-1}) + (X_i - x_{i-1}) F'(X_i),$$

$$k(x, t, u(t))|_{x \in X_i, t \in X_l} \in K^1_{ij} := k(x_{i-1}, t_{l-1}, \underline{u_0}) + (X_i - x_{i-1}) K_x(X_i, X_l, U_0) + (X_l - x_{l-1}) K_t(X_i, X_l, U_0),$$

and

$$f(x)|_{x \in X_i} \in F^2_i := f(x_{i-1}) + (X_i - x_{i-1}) f'(x_{i-1}) + \frac{1}{2}(X_i - x_{i-1})^2 F''(X_i),$$

$$k(x, t, u(t))|_{x \in X_i, t \in X_l} \in K^2_{ij} := k(x_{i-1}, t_{l-1}, \underline{u_0}) + (X_i - x_{i-1}) \frac{d}{dx} k(x_{i-1}, t_{l-1}, \underline{u_0}) + (X_l - x_{l-1}) \frac{d}{dt} k(x_{i-1}, t_{l-1}, \underline{u_0}) + \frac{1}{2}(X_i - x_{i-1})^2 K_{xx}(X_i, X_l, U_0) + \frac{1}{2}(X_l - x_{l-1})^2 K_{tt}(X_i, X_l, U_0) + (X_i - x_{i-1})(X_l - x_{l-1}) K_{xt}(X_i, X_l, U_0).$$

Then from (3.3) we get

$$F^1_i := f(x_{i-1}) + [0, w(X_i)] F'(X_i),$$

$$K^1_{ij} := k(x_{i-1}, t_{l-1}, \underline{u_0}) + [0, w(X_i)] K_x(X_i, X_l, U_0) + [0, w(X_l)] K_t(X_i, X_l, U_0),$$

and

$$f(x)|_{x \in X_i} \in F^2_i := f(x_{i-1}) + [0, w(X_i)] f'(x_{i-1}) + \frac{1}{2}[0, w(X_i)]^2 F''(X_i),$$

$$k(x, t, u(t))|_{x \in X_i, t \in X_l} \in K^2_{ij} := k(x_{i-1}, t_{l-1}, \underline{u_0}) + [0, w(X_i)] \frac{d}{dx} k(x_{i-1}, t_{l-1}, \underline{u_0}) + [0, w(X_l)] \frac{d}{dt} k(x_{i-1}, t_{l-1}, \underline{u_0}) + \frac{1}{2}[0, w(X_i)]^2 K_{xx}(X_i, X_l, U_0) + \frac{1}{2}[0, w(X_l)]^2 K_{tt}(X_i, X_l, U_0) + [0, w(X_i) w(X_l)] K_{xt}(X_i, X_l, U_0).$$

Substituting F^1_i, K^1_{ij}, F^2_i and K^2_{ij} for F, K in (3.4), we obtain $U^1_{1,i}$ and $U^2_{1,i}$, respectively as new enclosures for (3.2). □

From this standpoint, we study an important property of these new enclosures:

Theorem 2. *The width of interval enclosures* $U^{\gamma}_{1,i},\ \gamma = 1,2$ *in* (3.5) *tends to zero as the number of subintervals increases.*

Proof. Since f and k and their derivatives are continuous on closed subintervals, so they are bounded and attain their maximum and minimum. Here c_i's denote the maximums and the widths of bounded real and interval functions and C_i's denote constants. Let $\delta = w(X_i) = \frac{\chi}{p},\ i = 1,\ldots,p$, where p is the number of subdivisions, then by the definition of $F^1_i,\ K^1_{ij}$ and using (2.1) and (2.2), we may write

$$w(F^1_i) = w([0,w(X_i)]F'(X_i)) \leqslant |[0,w(X_i)]|w(F'(X_i)) + w([0,w(X_i)])|F'(X_i)| \leqslant \delta c_1 + \delta c_2 = \delta(c_1 + c_2) = \delta C_1, \tag{3.6}$$

$$\begin{aligned} w(K^1_{ij}) &= w([0,w(X_i)]K_x(X_i,X_l,U_0)) + w([0,w(X_l)]K_t(X_i,X_l,U_0)) \\ &\leqslant |[0,w(X_i)]|w(K_x(X_i,X_l,U_0)) + w([0,w(X_i)])|K_x(X_i,X_l,U_0)| + |[0,w(X_l)]|w(K_t(X_i,X_l,U_0))) \\ &\quad + w([0,w(X_l)])|K_t(X_i,X_l,U_0))| \\ &\leqslant \delta(c_3 + c_4 + c_5 + c_6) = \delta C_2. \end{aligned} \tag{3.7}$$

Similar results can be obtained for F^2_i and K^2_{ij}:

$$\begin{aligned} &w\left(F^2_i\right) \leqslant \delta c_7 + \frac{1}{2}\left(\delta^2 c_8 + \delta^2 c_9\right) = \delta C_3, \\ &w\left(K^2_{ij}\right) \leqslant \delta(c_{10} + c_{11}) + \delta^2 \sum_{i=12}^{15} \frac{1}{2} c_i + \delta^2(c_{16} + c_{17}) = \delta C_4. \end{aligned} \tag{3.8}$$

Also from (3.5), for $\gamma = 1,2$, we have

$$w(U^{\gamma}_{1,i}) \leqslant w(F^{\gamma}_i) + \sum_{l=1}^{i-1} w(K^{\gamma}_{ij} w(X_l)) + w(K^{\gamma}_{ii}[0,w(X_i)]) \leqslant w(F^{\gamma}_i) + \delta \sum_{l=1}^{i} w(K^{\gamma}_{ij}) + \delta |K^{\gamma}_{ii}|.$$

Substituting (3.6), (3.7) and (3.8) into the above relation, we get

$$\begin{aligned} &w(U^1_{1,i}) \leqslant \delta C_1 + \delta^2 \sum_{l=1}^{i} C_2 + \delta^2 C_5 = \delta C_7, \\ &w(U^2_{1,i}) \leqslant \delta C_3 + \delta^2 \sum_{l=1}^{i} C_4 + \delta^2 C_6 = \delta C_8. \end{aligned}$$

As $\delta = \frac{\chi}{p}$, we conclude that

$$\lim_{p\to\infty} \delta = 0,$$

so $w(U^1_{1,i}) \to 0$ and $w(U^2_{1,i}) \to 0$ by increasing the number of subdivisions and this completes the proof. □

Now, we substitute U_0 with $U^{\gamma}_{1,i},\ \gamma = 1,2$ in (3.5), and get the next approximation $U^{\gamma}_{2,i},\ \gamma = 1,2$. This process can be done continually yielding a nested sequence of intervals with the following property:

Corollary 1. *The iterated sequences of intervals using the new enclosure introduced in* (3.5) *has at least superlinear rate of convergence i.e.* $U^{\gamma}_{n,i} = \mathcal{O}(\delta)$, *with* $\gamma = 1,2$.

3.3. Practical computation of partial derivatives

In order to construct an algorithm for the method, we need to calculate the partial derivatives of $K_x(X_i,X_l,U_0)$, $K_t(X_i,X_l,U_0)$ and $K_{xt}(X_i,X_l,U_0)$. We have

$$\frac{d}{dt}k(x,t,u(t)) = \frac{\partial}{\partial t}k(x,t,u(t)) + \frac{\partial}{\partial u}k(x,t,u(t))u'(t),$$

where $\frac{\partial}{\partial t}$ is a partial derivative respect to t. In order to compute the derivatives of the unknown function $u(t)$ respect to t, we need initial enclosures for $u'(t)$ and $u''(t)$, namely U'_0 and U''_0. Hence differentiating equation (1.1) yields

$$u'(t) = f'(t) + \frac{d}{dt}\int_0^t k(t,s,u(s))ds = f'(t) + \int_0^t \frac{\partial}{\partial t}k(t,s,u(s))ds + k(t,t,u(t)). \tag{3.9}$$

Let $K_{(t)}(X_i, X_i, U_0) := \{\frac{\partial}{\partial t} k(x, t, u(t)) | x, t \in X_i, u(t) \in U_0\}$. From the concept of interval integration, the following enclosure can be obtained

$$u'(x)|_{x \in X_i} \in U'_{1,i} := F'(X_i) + K_{(t)}(X_i, X_i, U_0)[0, w(T_i)] + K(X_i, X_i, U_0).$$

Consequently if $K_{(tt)}(X_i, X_i, U_0) := \{\frac{\partial^2}{\partial t^2} k(x, t, u(t)) | x, t \in X_i, u(t) \in U_0\}$, then the interval extension of the second derivatives can be easily obtained as follows:

$$u''(x)|_{x \in X_i} \in U''_{1,i} := F''(X_i) + K_{(tt)}(X_i, X_i, U_0)[0, w(T_i)] + 2K_{(t)}(X_i, X_i, U_0) + K_{(s)}(X_i, X_i, U_0) + K_{(u)}(X_i, X_i, U_0)U'_0. \quad (3.10)$$

Similarly, other derivatives and enclosures for the next iteration can be obtained in a similar manner.

Using these notations, we can summarize the procedures in the following algorithm:

Algorithm 2. First and second order approximation

Input:
  Number of subdivisions p;
  Number of iterations n;
  U_0;
begin
  For $k = 0, \ldots, n$:
    For $i = 0, \ldots, p-1$:
      Compute $U^{\gamma}_{1,i},\ \gamma = 1, 2$. from (3.5);
      Compute $U'_{1,i}$ and $U''_{1,i}$ from (3.9) and (3.10);
      $U_0 \leftarrow U_{1,i} \cap U_0, U'_0 \leftarrow U'_{1,i} \cap U'_0,\ U''_0 \leftarrow U''_{1,i} \cap U''_0$;
$\forall i,\ u(x_i) \in U_{1,i} \cap U_{1,i+1}$;
end.

Due to the properties of the modified algorithm like superlinear convergence, new enclosures $U^{\gamma}_{1,i},\ \gamma = 1, 2$ are considerably better in comparison to $U^{\gamma}_{0,i},\ \gamma = 1, 2$, especially when the number of subdivisions is increased, so in many cases just a few iterations (usually $n = 3$ as in our numerical experiments) is needed in practice and the method rapidly converges to the appropriate solutions. It is clear, the same process can be applied with more terms of Taylor expansion to achieve higher precision.

4. Numerical experiments and discussions

To illustrate the above algorithms, we consider three test problems taken from [29,30,15]. We apply zero, first and second order approximations to the examples so that comparisons can be made numerically with the existing results. All the computations are calculated by using the symbolic manipulation software Mathematica®.

$$\begin{aligned} &u(x) = -x^3(-1 + e^{\sin x}) + \sin x + \int_0^x x^3 \cos t e^{u(t)} dt, \quad x \in [0, 1], \\ &u(x) = \sin x, \end{aligned} \quad (4.1)$$

$$\begin{aligned} &u(x) = e^x - \frac{1}{3} e^{3x} + \frac{1}{3} + \int_0^x u^3(t) dt, \quad x \in [0, 1], \\ &u(x) = e^x, \end{aligned} \quad (4.2)$$

$$\begin{aligned} &\begin{cases} u'(x) = 1 + \frac{x}{2} - e^x - \frac{x e^{-x^2}}{2} + e^{u(x)} - \int_0^x x t e^{-u^2(t)} dt, \quad x \in [0, 1], \\ u(0) = 0, \end{cases} \\ &u(x) = x. \end{aligned} \quad (4.3)$$

We select $U_0 = [0, 1]$ in the test problems (4.1), (4.3) and $U_0 = [1, 2]$ in the problem (4.2), respectively. The solutions of Eq. (4.1) for 100 subdivisions based on the second order scheme is summarized in Tables 1 and 2 shows the results of numerical verification for *three* iterations in $[0, 0.5]$:

It can easily be verified that the exact solutions are enclosed in interval results of the algorithm. The diameter of interval solutions is increasing due to the accumulation of truncation and round off errors. Despite this fact, the results of our verification algorithm is better than non-guaranteed methods such as those given in [29]. Eq. (4.1) was solved in [29] using eight terms of the classical Taylor polynomials ($N = 8$) and the maximum absolute error is $\mathcal{O}(10^{-5})$ in $[0, 0.5]$.

Note that, the maximum error of the proposed method for the zero order algorithm is not affected by the number of subdivisions but the first and second order methods are completely dependent on the number of divisions. Also as we pointed out, Taylor polynomials of higher order can be applied to improve the numerical results.

Consequently, the solutions of the test problem (4.2) will be approximated by the algorithm for three iterations in [0, 0.5] and the maximum absolute errors are given in Table 3.

The classical Taylor polynomial method in [15], gives the maximum absolute error $\mathcal{O}(10^{-6})$ for $N = 6$ in [0, 0.5]. Looking at Table 3, we can observe that there is an improvement in the accuracy for $N = 2$ compared to the method given in [15].

Finally, the results of the method for the integro–differential equation (4.3) with three iterations in [0, 0.5] is illustrated in Table 4:

Table 1
Numerical results for Eq. (4.1) for $n = 3$ and $p = 100$.

x_i	Exact solution	Results of Alg. 2	Diam of the interval
0.01	0.00999983	[0.00999983, 0.00999991]	8.52×10^{-8}
0.02	0.01999866	[0.01999866, 0.01999889]	2.25×10^{-7}
0.03	0.02999550	[0.02999549, 0.02999588]	3.81×10^{-7}
0.04	0.03998933	[0.03998932, 0.03998988]	5.54×10^{-7}
0.05	0.04997916	[0.04997915, 0.04997989]	7.44×10^{-7}
⋮	⋮	⋮	⋮

Table 2
Width of interval results of (4.1) for $n = 3$ and $x \in [0, 0.5]$.

Order (N)	Divisions (p)	Max error
0	50	6.78×10^{-4}
	100	6.66×10^{-4}
	200	8.22×10^{-4}
	400	6.53×10^{-4}
1	50	3.79×10^{-5}
	100	5.70×10^{-5}
	200	5.38×10^{-6}
	400	2.68×10^{-6}
2	50	2.61×10^{-5}
	100	4.60×10^{-6}
	200	7.72×10^{-7}
	400	9.08×10^{-8}

Table 3
Width of interval results of (4.2) for $n = 3$ and $x \in [0, 0.5]$.

Order (N)	Divisions (p)	Max error
0	50	1.91×10^{-2}
	100	7.19×10^{-2}
	200	1.21×10^{-2}
	400	1.96×10^{-2}
1	50	1.31×10^{-2}
	100	9.52×10^{-3}
	200	3.05×10^{-3}
	400	9.72×10^{-4}
2	50	4.59×10^{-4}
	100	4.60×10^{-5}
	200	3.09×10^{-6}
	400	9.08×10^{-7}

Table 4
Width of interval results of (4.3) for $n = 3$ and $x \in [0, 0.5]$.

Order (N)	Divisions (p)	Max error
0	50	4.60×10^{-2}
	100	3.25×10^{-2}
	200	3.82×10^{-2}
	400	4.24×10^{-2}
1	50	7.76×10^{-3}
	100	4.36×10^{-3}
	200	1.72×10^{-3}
	400	6.61×10^{-4}
2	50	5.03×10^{-3}
	100	1.77×10^{-3}
	200	7.16×10^{-4}
	400	2.48×10^{-4}

The maximum error bounds of the collocation method in [30] are $\mathcal{O}(10^{-4})$ and $\mathcal{O}(10^{-6})$ for $N = 4,16$, respectively. The results of [30] seems to be better in comparison to our results but it should be noted that the interval enclosure algorithms of this paper are able to compute guaranteed error bounds including all discretization and round-off errors in comparison with most classical numerical methods. Despite this situation, more terms of Taylor expansion or other efficient methods e.g. Taylor model can be applied to achieve higher precision which will be the subject of future works.

5. Conclusion

In this paper, a verification method was applied to nonlinear Volterra integral equations which enables us to compute guaranteed error bounds for the equation. We use interval polynomial enclosures using truncated Taylor series together with bounded remainders to increase the accuracy of the scheme while reducing the width of interval solutions. We have also compared the efficiency of our algorithm with classical numerical methods.

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