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Jacobi spectral solution for integral algebraic equations of index-2

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ABSTRACT

This paper is concerned with obtaining the approximate solution of a class of semi-explicit Integral Algebraic Equations (**IAEs**) of index-2. A Jacobi collocation method including the matrix–vector multiplication representation is proposed for the IAEs of index-2. A rigorous analysis of error bound in weighted L^2 norm is also provided which theoretically justifies the spectral rate of convergence while the kernels and the source functions are sufficiently smooth. Results of several numerical experiments are presented which support the theoretical results.

1. Introduction

Coupled systems of Integral Algebraic Equations (**IAEs**) consisting of the first and second kind Volterra Integral Equations (**VIEs**) naturally arise in many mathematical modeling processes, e.g. the kernel identification problems in heat conduction and viscoelasticity [21], evolution of a chemical reaction within a small cell [14], the two dimensional biharmonic equation in a semi-infinite strip [10], dynamic processes in chemical reactors [15] and Kirchhoff's laws. (For further applications see [2,22] and references therein.)

An initial investigation of these equations indicates that they have properties very similar to Differential Algebraic Equations (**DAEs**). A system of DAEs is characterized by its index, which is the number of differentiations required to convert it into a system of ODEs. The concept of "index" has been introduced in order to quantify the level of difficulty that is involved in solving a given DAE or IAE (see e.g. [1,12,13]). It must be stressed that the numerical schemes which applicable (i.e. convergent) for IAEs of a given index, might not be useful for IAEs of higher index. Note that IAEs with index > 1 are generally hard to solve and are still under active research.

The theory of IAEs appeared from early attempts by Gear in the 1990 that determined the difficulties of these equations. He introduced the "index reduction procedure" for IAEs system in [8] similar to that in [9] for DAEs in which if the process is terminated, then the index is determined. This means that under suitable conditions, there is a solution for the resulting regular system of integral equations.

Since then, several authors have investigated the existence, uniqueness and numerical analysis of IAEs systems. Bulatov [3] in 1997, gave the existence and uniqueness results of solution for IAEs systems with convolutions kernels and defined the index in analogy to Gear's approach. (See [4] for further details.) Kauthen [16] in 2000, applied the polynomial spline collocation method for a semi-explicit IAEs with index-1 and established global convergence as well as local superconvergence. Furthermore, Brunner [2] defined the index-1 tractable for a semi-explicit form of IAEs and investigated the existence of a unique solution for these types of systems.

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In this paper, we study the numerical solvability of a mixed system of Volterra integral equations of the first and second kind. More precisely, we consider the semi-explicit system of integral algebraic equations

$$\begin{cases} y(t) = f(t) + (\nu_{11} y)(t) + (\nu_{12} z)(t), \\ 0 = g(t) + (\nu_{21} y)(t), \end{cases} \tag{1.1}$$

where the linear Volterra integral operators ν_{kl} are given by:

$$(\nu_{kl}\varphi)(t) = \int_0^t K_{kl}(t,s)\varphi(s)\,ds, \quad t \in I = [0,T] \quad (k,l = 1,2)$$

and $y : I \to \mathbb{R}^{d_1}$, $z : I \to \mathbb{R}^{d_2}$, $K_{11}(\cdot,\cdot) \in L(\mathbb{R}^{d_1})$, $K_{12}(\cdot,\cdot) \in L(\mathbb{R}^{d_2}, \mathbb{R}^{d_1})$, $K_{21}(\cdot,\cdot) \in L(\mathbb{R}^{d_1}, \mathbb{R}^{d_2})$ and $L(\cdot,\cdot)$ is the linear transformation space. Here, the word "algebraic" assumes a wider meaning, in that it refers to the "non-differential" constraints forming part of the system in analogy to DAEs.

In order to give an application of these models, consider the following heat equation with initial and mixed boundary conditions which represents a boundary reaction in diffusion of chemicals where $\alpha(t)u_x$ represents the diffusive transport of materials to the boundary. Following [5, p. 79], for continuously differentiable function f and for continuous functions g, h, α, β, and γ, the solution of the problem

$$\begin{cases} u_t = u_{xx}, \quad 0 < x < 1, \quad 0 < t, \\ u(x,0) = f(x), \quad 0 < x < 1, \\ u_t(0,t) + \alpha(t)u_x(0,t) + \beta(t)u(0,t) = g(t), \quad 0 < t, \\ u_x(1,t) + \gamma(t)u(1,t) = h(t), \quad 0 < t, \end{cases} \tag{1.2}$$

has the representation

$$\begin{aligned} u(x,t) = {} & \int_0^1 \{\theta(x-\xi,t) - \theta(x+\xi,t)\} f(\xi)\,d\xi - 2\int_0^t \frac{\partial\theta}{\partial x}(x,t-\tau)\left\{ f(0) + \int_0^\tau \phi_1(\eta)\,d\eta \right\} d\tau \\ & + 2\int_0^t \frac{\partial\theta}{\partial x}(x-1,t-\tau)\left\{ f(1) + \int_0^\tau \phi_2(\eta)\,d\eta \right\} d\tau, \end{aligned} \tag{1.3}$$

if and only if ϕ_1 and ϕ_2 are continuous functions that satisfy in the coupled system of Volterra integral equations

$$\begin{cases} g(t) = \phi_1(t) + 2\alpha(t)\displaystyle\int_0^t \theta(\xi,t) f'(\xi)\,d\xi - 2\alpha(t)\int_0^t \theta(0,t-\tau)\phi_1(\tau)\,d\tau \\ \qquad + 2\alpha(t)\displaystyle\int_0^t \theta(-1,t-\tau)\phi_2(\tau)\,d\tau + \beta(t)f(0) + \beta(t)\int_0^t \phi_1(\tau)\,d\tau, \\ h(t) = 2\displaystyle\int_0^t \theta(1+\xi,t) f'(\xi)\,d\xi - 2\int_0^t \theta(1,t-\tau)\phi_1(\tau)\,d\tau \\ \qquad + 2\displaystyle\int_0^t \theta(0,t-\tau)\phi_2(\tau)\,d\tau + \gamma(t)f(1) + \gamma(t)\int_0^t \phi_2(\tau)\,d\tau, \end{cases} \tag{1.4}$$

where $\theta(x,t)$ is a well-known Theta function which is given in [5].

However, in some special cases (e.g. $\gamma(t) = \frac{-2\int_0^t \theta(0,t-\tau)\phi_2(\tau)\,d\tau}{f(1)+\int_0^t \phi_2(\tau)\,d\tau}$, $u(1,t) = 0$), it is possible to reduce the system (1.4) to a system of the form (1.1) as follows:

$$\begin{cases} g(t) = \phi_1(t) + 2\alpha(t) \int_0^t \theta(\xi, t) f'(\xi)\, d\xi - 2\alpha(t) \int_0^t \theta(0, t-\tau)\phi_1(\tau)\, d\tau \\ \qquad + 2\alpha(t) \int_0^t \theta(-1, t-\tau)\phi_2(\tau)\, d\tau + \beta(t) f(0) + \beta(t) \int_0^t \phi_1(\tau)\, d\tau, \\ h(t) = 2 \int_0^t \theta(1+\xi, t) f'(\xi)\, d\xi - 2 \int_0^t \theta(1, t-\tau)\phi_1(\tau)\, d\tau. \end{cases}$$

The outline of this paper is as follows: Section 2 is devoted to the introduce of IAEs and its preliminary concepts. In Section 3, we investigate the existence of a unique solution for the semi-explicit IAEs system (1.1). The numerical analysis and the error estimation of the Jacobi collocation method in weighted L^2 norm are given in Sections 4 and 5. Finally, in Section 6 some numerical experiments are reported which confirm the theoretical results of the paper.

2. Preliminaries

A general integral algebraic equations system takes the form:

$$A(t)X(t) = G(t) + \int_0^t K\big(t, s, X(s)\big)\, ds, \tag{2.1}$$

where $X : I \to \mathbb{R}^d$, $G : I \to \mathbb{R}^d$, $K : I \times I \times \mathbb{R}^d \to \mathbb{R}^d$ are continuous and $A(t) \in L(\mathbb{R}^d)$ is singular matrix with continuous entries (rank(A)$\geqslant 1$, $\det(A) = 0$). Note that since A is singular and the system has a structure similar to DAEs of index greater than zero, there is no guarantee that the above system is solvable. (See e.g. [8,9].)

The semi-explicit linear version of (2.1) is as follows:

$$\begin{cases} y(t) = f(t) + (\nu_{11} y)(t) + (\nu_{12} z)(t), \\ 0 = g(t) + (\nu_{21} y)(t) + (\nu_{22} z)(t), \end{cases} \tag{2.2}$$

where the Volterra integral operators ν_{kl} and the matrix kernels K_{kl} $(k, l = 1, 2)$, are defined as (1.1).

The notion of index is crucial for the behavior of solutions of IAEs and it must contain the properties of feasible numerical solutions. There exist several different definitions of index for IAEs in [2,4,8] which are often closely related. These definitions are conceptually based on the "index reduction procedure" i.e. a differentiation process of the algebraic constraints which yields a system of regular VIEs and the "tractability index" that is the algebraic constraints which are locally solvable for the algebraic components of the IAEs solution. The following definition regarding to index-1 tractable for the semi-explicit system (2.2) was given by Brunner which is analogous to that defined for DAEs by März [17]:

Definition 1. (See [2].) The semi-explicit IAEs (2.2), is said to be index-1 tractable if the first-kind VIE:

$$\int_0^t K_{22}(t, s) w(s)\, ds = h(t), \quad t \in I,$$

is uniquely solvable in $C(I)$, whenever $h \in C^1(I)$ and $h(0) = 0$.

This means that IAEs system (2.2) is index-1 tractable if its second equation (e.g. the linear first kind VIE) is uniquely solvable for $z \in C(I)$ as an algebraic component. Following [2], assume that the functions f, g, K_{ij} $(i, j = 1, 2)$ are sufficiently smooth and for all $t \in I$, $|\det K_{22}(t, t)| \geqslant k_0 > 0$, where k_0 is a positive constant and $g(0) = 0$. Using the index reduction procedure [8] and Theorem (2.1.8) in [2, p. 64], we can show that the linear VIE possesses a unique solution on I and so the solvability and regularity of its solution have been obtained.

3. IAEs of index-2

As a consequence of the previous section, it is possible to define the index-2 for the semi-explicit IAEs system (1.1) based on minimal regularity conditions and investigate the existence of a unique solution.

Considering the system (1.1) and differentiating from its second equation with respect to t, we obtain:

$$0 = g'(t) + K_{21}(t, t) y(t) + \int_0^t \frac{\partial K_{21}(t, s)}{\partial t} y(s)\, ds.$$

Now, by replacing $y(t)$ from the first equation of (1.1):

$$0 = g_1(t) + \int_0^t \left(K_{21}(t,t)K_{11}(t,s) + \frac{\partial K_{21}(t,s)}{\partial t} \right) y(s)\,ds + \int_0^t K_{21}(t,t)K_{12}(t,s)z(s)\,ds, \tag{3.1}$$

where $g_1(t) = g'(t) + K_{21}(t,t)f(t)$.

Obviously, (3.1) together with the first equation of (1.1) are the same as the index-1 semi-explicit IAEs system (2.2). In accordance with the terminology introduced by Gear [8], we will refer to the system (1.1) as a semi-explicit IAEs of index-2. However it has to be pointed out that this reduction to IAEs of index-1 (2.2) is not practical from a numerical point of view.

We can focus again on the tractability index-2 for the IAEs (1.1), in other view point. The IAEs (1.1) is called index-2 tractable, if the algebraic constraints are locally solvable for the algebraic components z i.e. by replacing $y(t)$ from the first equation of (1.1) into its second equation we have

$$0 = g(t) + f_1(t) + \int_0^t \int_0^s K_{21}(t,s)K_{12}(s,x)z(x)\,dx\,ds, \tag{3.2}$$

where

$$f_1(t) = \int_0^t K_{21}(t,s)f(s)\,ds + \int_0^t \int_0^s K_{21}(t,s)K_{11}(s,x)y(x)\,dx\,ds.$$

Exchanging the two integrals in Eq. (3.2) with respect to s, then x, we obtain:

$$0 = g(t) + f_1(t) + \int_0^t \overline{K}(t,x)z(x)\,dx, \tag{3.3}$$

where $\overline{K}(t,x) = \int_x^t K_{21}(t,s)K_{12}(s,x)\,ds$.

One of our result in this section is given in the following definition which gives the tractability index-2 for the semi-explicit IAEs system (1.1) using (3.3) and the extension of Definition 1:

Definition 2. The semi-explicit IAEs (1.1), is said to be index-2 tractable if the first-kind VIE:

$$\int_0^t \overline{K}(t,x)w(x)\,dx = h(t), \quad t \in I,$$

is uniquely solvable in $C(I)$ whenever $h \in C^1(I)$ and $h(0) = 0$.

Now, we will make use of the result of Brunner [2], which gives the existence and uniqueness results for IAEs (1.1). So, the VIE (3.1) is uniquely solvable for $z \in C(I)$, if the given functions $f, g, K_{ij}(i, j = 1, 2)$ and $\frac{\partial K_{21}(t,s)}{\partial t}$ are sufficiently smooth and

$$g_1(0) = 0, \quad \text{and} \quad \left|\det\bigl(K_{21}(t,t)K_{12}(t,t)\bigr)\right| \geqslant k_0 > 0, \quad \forall t \in I$$

where k_0 is a positive constant.

The following theorem gives the relevant conditions concerning K_{1l} $(l = 1, 2)$, K_{21}, f and g for the investigation of the unique solution of IAEs (1.1) analogous to Theorem (8.1.5) in [2, p. 472]:

Theorem 1. *Let $\nu \geqslant 0$ and assume that*

1. $K_{1l} \in C^\nu(D)$ *for* $l = 1, 2$ *and* $D = I \times I$,
2. $K_{21} \in C^{\nu+1}(D)$ *and* $|\det(K_{21}(t,t)K_{12}(t,t))| \geqslant k_0 > 0$,
3. $f \in C^\nu(D)$, $g \in C^{\nu+1}(D)$ *and* $g_1(0) = 0$, *then the IAEs* (1.1) *possesses a unique solution* $y, z \in C^\nu(I)$.

It is also possible to obtain the desired information on the regularity of the solutions in accordance to Theorem (8.1.6) of [2, p. 473]:

Theorem 2. *Assume that the hypotheses given in Theorem 1 hold with $\nu \geqslant 0$, then for $K_{21}(t,t)K_{11}(t,s) + \frac{\partial K_{21}(t,s)}{\partial t} = 0$ on D, the unique solution of the IAEs (1.1) is given by the representation*

$$y(t) = f(t) + \int_0^t R_{11}(t,s) f(s)\,ds + k_{12}(t) g_1(t) + \int_0^t Q_{12}(t,s) g_1(s)\,ds,$$

$$z(t) = k_{21}(t) g_1(t) - k^{-1} g_1'(t) + \int_0^t Q_{22}(t,s) g_1(s)\,ds,$$

where

$$k_{21} = -\frac{R_{22}(t,t)}{k}, \qquad k_{12} = -\left(\frac{2R_{11}(t,t) - R_{22}(t,t)}{k}\right),$$
$$Q_{22} = \frac{\partial}{\partial s}\left(\frac{R_{22}(t,s)}{k}\right), \qquad Q_{12} = \frac{\partial}{\partial s}\left(\frac{2R_{11}(t,s) - R_{22}(t,s)}{k}\right),$$

and $R_{11}(t,s)$ denotes the resolvent kernels associated with the kernel $K_{11}(t,s)$ in (1.1).

Proof. The proof is mainly based on the proof of the corresponding theorem in [2] and we refrain from going into details. It is sufficient we take $k = K_{21}(t,t)K_{12}(t,t)$ and $H_{22}(t,s) = -k^{-1}\frac{\partial}{\partial t}(K_{21}(t,t)K_{12}(t,s))$. Also, $R_{22}(t,s)$ is the resolvent kernels associated with the kernel $H_{22}(t,s)$. Under these conditions, the results of the theorem will be obtained. □

For considering the stability issue of the problem, we use the results of Gear [8], who has defined the index of IAEs by considering the effect of perturbations of the equations on the solutions and obtained the relationship between the differential and the perturbation index in which they are identical.

Theorem 2 reveals us the stability issue in the sense of perturbation index as follows:

The IAEs system (1.1) can be written as compact form

$$\mathbf{B}\mathrm{Y} = \mathrm{G}, \tag{3.4}$$

where $\mathrm{Y} = (y,z)^T$, $\mathrm{G} = (f,g)^T$ and $\mathbf{B} = \begin{pmatrix} I-\nu_{11} & -\nu_{12} \\ -\nu_{21} & 0 \end{pmatrix}$.

The corresponding perturbed system may be considered as

$$\mathbf{B}\tilde{\mathrm{Y}} = \mathrm{G} + \delta,$$

with $\delta = (\delta_1, \delta_2)^T$.

Differentiating the second equation of the perturbed system with respect to t and substituting $\tilde{y}$ from the first equation, yields

$$\mathbf{B_1}\tilde{\mathrm{Y}} = \mathrm{G}_1 + \delta', \tag{3.5}$$

where $\mathbf{B_1} = \begin{pmatrix} I-\nu_{11} & -\nu_{12} \\ -\mathcal{V}_1 & -\mathcal{V}_2 \end{pmatrix}$, $\mathcal{V}_i = \int_0^t \mathcal{K}_i(t,s)\,ds$, $i = 1,2$, with

$$\mathcal{K}_1(t,s) = K_{21}(t,t)K_{11}(t,s) + \frac{\partial K_{21}(t,s)}{\partial t}, \qquad \mathcal{K}_2(t,s) = K_{21}(t,t)K_{12}(t,s)$$

and

$$\delta' = \left(\delta_1, K_{21}(t,t)\delta_1 + \delta_2'\right)^T, \qquad \mathrm{G}_1 = (f, g_1)^T \quad \text{with } g_1(t) = g'(t) + K_{21}(t,t) f(t).$$

Also, differentiating the second equation of (3.5) with respect to t, gives

$$\mathbf{B_2}\tilde{\mathrm{Y}} = \mathrm{G}_2 + \delta'',$$

where

$$\mathbf{B_2} = \begin{pmatrix} I-\nu_{11} & -\nu_{12} \\ -\mathcal{V}_1' & -K_{21}(t,t)K_{12}(t,t) - \mathcal{V}_2' \end{pmatrix},$$

$$\mathcal{V}_i' = \int_0^t \frac{\partial}{\partial t}\mathcal{K}_i(t,s)\,ds, \qquad \mathrm{G}_2 = (f, g_2)^T, \qquad \delta'' = \left(\delta_1, K_{21}'(t,t)\delta_1 + K_{21}(t,t)\delta_1' + \delta_2''\right)^T,$$

and

$$g_2(t) = g_1'(t) + \left(K_{21}(t,t)K_{11}(t,t) + \frac{\partial K_{21}(t,t)}{\partial t}\right)\tilde{y}(t).$$

Furthermore, for the original system (3.4) we have

$$\mathbf{B_2}\mathrm{Y} = \mathrm{G}_2.$$

Applying Theorem 2, we can deduce that the operator $\mathbf{B_2}$ is a linear, bounded and bijective such that $\mathbf{B_2^{-1}}$ exists and is bounded. Then it follows from the above relations

$$\|\mathrm{Y} - \tilde{\mathrm{Y}}\| \leqslant C\|\mathbf{B_2^{-1}}\|\left(\|\delta_1\| + \left\|\delta_1'\right\| + \left\|\delta_2''\right\|\right),$$

where C is a constant independent of t.

This yields the stability estimate of the IAEs system (1.1) in the sense of the perturbation index.

4. The Jacobi collocation scheme

This section is devoted to applying the Jacobi collocation method to numerically solve the IAEs system of index-2. To do so, we consider a collocation method including the matrix–vector multiplication representation of the equations.

Let $\mathcal{P}_N(\Lambda)$ be the space of all polynomials with degree not exceeding N on Λ, where Λ stands for the open interval $(-1,1)$ and $w^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$, for $\alpha, \beta > -1$ denotes a weight function in the usual sense. It is well known, the set of Jacobi polynomials $\{J_N^{\alpha,\beta}\}_{N=0}^{\infty}$ forms a complete $L^2_{w^{\alpha,\beta}}(\Lambda)$ orthogonal system, where $L^2_{w^{\alpha,\beta}}(\Lambda)$ is the space of functions $f:[-1,1]\to\mathbb{R}$ with $\|f\|^2_{L^2_{w^{\alpha,\beta}}} < \infty$, and

$$\|f\|^2_{L^2_{w^{\alpha,\beta}}} = \langle f, f\rangle_{L^2_{w^{\alpha,\beta}}} = \int_{-1}^{1} |f(x)|^2 w^{\alpha,\beta}(x)\,dx.$$

So, any function $u(x)$ in the space $L^2_{w^{\alpha,\beta}}(\Lambda)$ admits the expansion $u(x) = \sum_{N=0}^{\infty}\tilde{u}_N J_N^{\alpha,\beta}$ with

$$\tilde{u}_N = \frac{1}{\|J_k^{\alpha,\beta}\|^2_{L^2_{w^{\alpha,\beta}}}}\int_{-1}^{1} u(\xi)J_N^{\alpha,\beta}(\xi)w^{\alpha,\beta}(\xi)\,d\xi.$$

Now, let $H^m_{w^{\alpha,\beta}}$ denotes the Sobolev space of all functions $u(x)$ on Λ such that $u(x)$ and all its weak derivatives up to order m are in $L^2_{w^{\alpha,\beta}}(\Lambda)$, with the norm and the semi-norm as

$$\begin{aligned}
\left\|u(x)\right\|^2_{H^m_{w^{\alpha,\beta}}(\Lambda)} &= \sum_{k=0}^{m}\left\|\frac{\partial^k}{\partial x^k}u(x)\right\|^2_{L^2_{w^{\alpha,\beta}}(\Lambda)}, \\
|u|_{H^{m,N}_{w^{\alpha,\beta}}(\Lambda)} &= \left(\sum_{j=\min(m,N+1)}^{m}\left\|u^{(j)}\right\|^2_{L^2_{w^{\alpha,\beta}}(\Lambda)}\right)^{1/2}.
\end{aligned} \tag{4.1}$$

Let Π_N be the orthogonal projective operator from $L^2_{w^{\alpha,\beta}}(\Lambda)$ on to $\mathcal{P}_N(\Lambda)$. It means that for any function φ in $L^2_{w^{\alpha,\beta}}(\Lambda)$, $\Pi_N\varphi$ belongs to $\mathcal{P}_N(\Lambda)$ and satisfies

$$\forall\psi_N \in \mathcal{P}_N(\Lambda), \quad \int_{-1}^{1}(\varphi - \Pi_N\varphi)(\xi)\psi_N(\xi)w^{\alpha,\beta}(\xi)\,d\xi = 0.$$

For any $u(x) \in C[-1,1]$, we can define the projection $I_N^{\alpha,\beta}$ as a Lagrange interpolating polynomial associated with the Jacobi polynomials (see e.g. [19])

$$I_N^{\alpha,\beta}u(x) = \sum_{i=0}^{N}\tilde{u}_i J_i^{\alpha,\beta}(x) = \sum_{i=0}^{N}u(x_i)L_i(x),$$

where the Lagrange polynomials $L_i(x)$ take the form

$$L_i(x) = w_i \sum_{k=0}^{N} \frac{1}{\tilde{\gamma}_k} J_k^{\alpha,\beta}(x_i) J_k^{\alpha,\beta}(x) \quad (i = 0, 1, \ldots, N)$$

and the collocation points $\{x_i\}_{i=0}^N$ represent the Jacobi–Gauss quadrature points used to compute the discrete expansion coefficients $\tilde{u}_k$,

$$\tilde{u}_k = \frac{1}{\tilde{\gamma}_k} \sum_{i=0}^{N} u(x_i) J_k^{\alpha,\beta}(x_i) w_i, \tag{4.2}$$

where $\tilde{\gamma}_k$ and w_i were given in [11, p. 231].

Let us now turn our attention toward the application of Jacobi collocation method for the following linear semi-explicit IAEs system:

$$A(t)X(t) = G(t) + \int_0^t K(t,s)X(s)\,ds, \quad t \in [0,T] \tag{4.3}$$

where $X : [0,T] \to \mathbb{R}^d$, $G : [0,T] \to \mathbb{R}^d$, $K \in L(\mathbb{R}^d)$, are continuous and $A(t) = \operatorname{diag}(\mathbf{I}_{d_1}, \mathbf{O}_{d_2}) \in L(\mathbb{R}^d)$ is a singular block matrix and $X(t) = \{x_i(t)\}_{i=1}^d$, $K(t,s) = \{k_{ij}(t,s)\}_{i,j=1}^d$.

To use the theory of orthogonal Jacobi polynomials, we consider the change of variables:

$$s = \frac{T}{2}(\eta + 1), \quad -1 \leqslant \eta \leqslant \tau, \qquad t = \frac{T}{2}(\tau + 1), \quad -1 \leqslant \tau \leqslant 1 \tag{4.4}$$

to rewrite the problem (4.3) as follows

$$\widehat{A}(\tau)\widehat{X}(\tau) = \widehat{G}(\tau) + \int_{-1}^{\tau} \widehat{K}(\tau,\eta)\widehat{X}(\eta)\,d\eta, \quad \tau \in [-1,1] \tag{4.5}$$

where

$$\widehat{G}(\tau) = G\left(\frac{T}{2}(\tau+1)\right), \qquad \widehat{X}(\tau) = X\left(\frac{T}{2}(\tau+1)\right), \qquad \widehat{K}(\tau,\eta) = \frac{T}{2} K\left(\frac{T}{2}(\tau+1), \frac{T}{2}(\eta+1)\right).$$

Approximating $\widehat{K}(\tau_m, \eta)$ with Jacobi polynomials gives

$$\widehat{K}_N(\tau_m,\eta) = \left\{\hat{k}_{ij}(\tau_m,\eta)\right\}_{i,j=1}^d = \left\{\sum_{k=0}^{N} (\hat{k}_{ij})_{mk} J_k^{\alpha,\beta}(\eta)\right\}_{i,j=1}^d = \left\{(\hat{\mathbf{k}}_{\mathbf{ij}})_{\mathbf{m}}\right\}_{i,j=1}^d \otimes \left(\mathbf{V}\mathbf{W}'\right), \tag{4.6}$$

where $\mathbf{V}$ is a lower triangular coefficient matrix of Jacobi polynomials with $\{J_i^{\alpha,\beta}(\eta)\}_{i=0}^N = \mathbf{V}\mathbf{W}'$, and

$$\mathbf{W}' = \left(1, \eta, \ldots, \eta^N\right)^T, \qquad (\hat{\mathbf{k}}_{\mathbf{ij}})_{\mathbf{m}} = \left((\hat{k}_{ij})_{m0}, \ldots, (\hat{k}_{ij})_{mN}\right),$$

$$(\hat{k}_{ij})_{mk} = \frac{1}{\|J_k^{\alpha,\beta}\|^2_{L^2_{w^{\alpha,\beta}}}} \int_{-1}^{1} \hat{k}_{ij}(\tau_m,\xi) J_k^{\alpha,\beta}(\xi) w^{\alpha,\beta}(\xi)\,d\xi \quad (k = 0, 1, \ldots, N)$$

and $\otimes$ is a Kronecker product.

Consequently, we seek a solution

$$I_N^{\alpha,\beta}\left(\widehat{X}(\eta)\right) = \widehat{X}_N(\eta) = \left\{\sum_{k=0}^{N} (\tilde{x}_i)_k J_k^{\alpha,\beta}(\eta)\right\}_{i=1}^d = \left\{\sum_{k=0}^{N} (\hat{x}_i)(\tau_k) L_k(\eta)\right\}_{i=1}^d = \{\hat{\mathbf{x}}_{\mathbf{i}}\}_{i=1}^d \otimes \left(\mathbf{V}'\mathbf{W}'\right), \tag{4.7}$$

where $\mathbf{V}' = \mathbf{J}\mathbf{V}$ with $\mathbf{J} = \{\frac{1}{\tilde{\gamma}_j} w_k J_j^{\alpha,\beta}(\tau_k)\}_{k,j=0}^N$, $\hat{\mathbf{x}}_{\mathbf{i}} = ((\hat{x}_i)(\tau_0), (\hat{x}_i)(\tau_1), \ldots, (\hat{x}_i)(\tau_N))$, $\{\tau_k\}_{k=0}^N$ are the Gauss–Jacobi collocation points (i.e. the zeros of $J_{N+1}^{\alpha,\beta}$), the coefficients $\{(\tilde{x}_i)_k\}_{k=0}^N$ are also given by (4.2) and $\{L_k\}_{k=0}^N$ are the interpolating Lagrange polynomials.

Inserting the collocation points $\{\tau_m\}_{m=0}^N$ in (4.5), we obtain the collocation equation as

$$\widehat{A}(\tau_m)\widehat{X}(\tau_m) = \widehat{G}(\tau_m) + \int_{-1}^{\tau_m} \widehat{K}_N(\tau_m,\eta)\widehat{X}_N(\eta)\,d\eta \quad (m = 0, 1, \ldots, N) \tag{4.8}$$

and using (4.6), (4.7) and (4.8), we get

$$\widehat{A}(\tau_m)\{\hat{\mathbf{x}}_{\mathbf{i}}\}_{i=1}^{d}=\widehat{G}(\tau_m)+\int_{-1}^{\tau_m}\big(\big\{(\hat{\mathbf{k}}_{\mathbf{ij}})_{\mathbf{m}}\big\}_{i,j=1}^{d}\otimes\big(\mathbf{VW}'\big)\big)\big(\{\hat{\mathbf{x}}_{\mathbf{i}}\}_{i=1}^{d}\otimes\big(\mathbf{V}'\mathbf{W}'\big)\big)\,d\eta\quad(m=0,1,\dots,N). \tag{4.9}$$

In this position, we give the following lemma, that will become instrumental in establishing the matrix vector multiplication representation of the product $\widehat{K}_N(\tau_m,\eta)\widehat{X}_N(\eta)$ in (4.8):

Lemma 1. *Let $P(\eta)=\sum_{i=0}^{N}p_i\eta^i$ and $Q(\eta)=\sum_{j=0}^{N}q_j\eta^j$ be two given polynomials, then*

$$P(\eta)Q(\eta)=\mathbf{P}(\mathbf{Q}\otimes\mathbf{M})\mathbf{W},$$

where $\mathbf{P}=(p_0,p_1,\dots,p_N)$, $\mathbf{Q}=(q_0,q_1,\dots,q_N)$, $\mathbf{W}=(1,\eta,\eta^2,\dots,\eta^{2N})^T$, and $\mathbf{M}$ is a block sparse matrix of the form $\mathbf{M}=(\mathbf{M}_N^{(k)})_{k=0}^{N}$ with

$$\mathbf{M}_j^{(k)}=\begin{pmatrix}\overbrace{0&\dots&0}^{k}&1&0&\dots&0&\overbrace{0&\dots&0}^{j-k}\\ 0&\dots&0&0&1&\dots&\vdots&0&\dots&0\\ \vdots&\dots&\vdots&\vdots&&\ddots&0&\vdots&\dots&\vdots\\ 0&\dots&0&\underbrace{0&\dots&0&1}_{N+1}&0&\dots&0\end{pmatrix}_{(N+1)(N+1+j)}.$$

Proof. Firstly, we will show that

$$\eta^m\sum_{i=0}^{N}p_i\eta^i=\mathbf{P}\mathbf{M}_m^{(m)}\mathcal{W}_m,$$

where $\mathcal{W}_m=(1,\eta,\eta^2,\dots,\eta^{N+m})^T$.

The validity of the equation for $m=0$ is obvious. Now we proceed by induction. So, we assume the validity of the equation for $m=k$ and transit to $m=k+1$ as follows

$$\eta^{k+1}\sum_{i=0}^{N}p_i\eta^i=\eta^k\sum_{i=0}^{N}(p_i\eta)\eta^i=\{p_i\eta\}_{i=0}^{N}\mathbf{M}_k^{(k)}\mathcal{W}_k=\mathbf{P}\mathbf{M}_k^{(k)}\big(\eta,\eta^2,\dots,\eta^{k+N+1}\big)^T.$$

With some simple manipulations we conclude

$$\mathbf{M}_k^{(k)}\big(\eta,\eta^2,\dots,\eta^{k+N+1}\big)^T=\mathbf{M}_{k+1}^{(k+1)}\big(1,\eta,\dots,\eta^{k+N+1}\big)^T=\mathbf{M}_{k+1}^{(k+1)}\mathcal{W}_{k+1}.$$

From these relations, we can write

$$\eta^{k+1}\sum_{i=0}^{N}p_i\eta^i=\mathbf{P}\mathbf{M}_{k+1}^{(k+1)}\mathcal{W}_{k+1},$$

and this proves the identity for $m=k+1$.

Now, consider the multiplication

$$P(\eta)Q(\eta)=\sum_{j=0}^{N}q_j\left(\sum_{i=0}^{N}p_i\eta^i\right)\eta^j=\sum_{j=0}^{N}q_j\mathbf{P}\mathbf{M}_j^{(j)}\mathcal{W}_j.$$

Due to the structure of $\mathbf{M}=(\mathbf{M}_N^{(k)})_{k=0}^{N}$, we have

$$\mathbf{M}_N^{(j)}\mathcal{W}_N=\mathbf{M}_j^{(j)}\mathcal{W}_j\quad(j=0,1,\dots,N).$$

This leads to the following relation

$$P(\eta)Q(\eta)=\mathbf{P}\left(\sum_{j=0}^{N}q_j\mathbf{M}_N^{(j)}\right)\mathcal{W}_N=\mathbf{P}(\mathbf{Q}\otimes\mathbf{M})\mathbf{W}.\qquad\square$$

Algorithm 1. The construction of the Jacobi collocation method including the matrix–vector multiplication representation of the equations

Step 1. Choose N, form the orthogonal bases $J_i^{\alpha,\beta}(\eta)$, $(i = 0, 1, \dots, N)$ and compute the nonsingular coefficient matrices $\mathbf{V}$ and $\mathbf{V}'$ as

$$\{J_i^{\alpha,\beta}(\eta)\}_{i=0}^{N} = \mathbf{V}\mathbf{W}',$$

and

$$\mathbf{V}' = \mathbf{J}\mathbf{V}$$

where $\mathbf{J} = \{\frac{1}{\gamma_j} w_k J_j^{\alpha,\beta}(\tau_k)\}_{k,j=0}^{N}$ and $\mathbf{W}' = (1, \eta, \dots, \eta^N)^T$.

Step 2. Compute the approximations of $\widehat{K}(\tau_m, \eta)$ and $\widehat{X}(\eta)$ using Jacobi polynomials from (4.6) and (4.7).

Step 3. Compute the matrix $\mathbf{M}$ from Lemma 1 and take $\mathbf{W_m} = \int_{-1}^{\tau_m} \mathbf{W}\, d\eta$ with $\mathbf{W} = (1, \eta, \eta^2, \dots, \eta^{2N})^T$.

Step 4. Solve the linear system (4.11) and obtain the entries of the vector solution $\{\hat{\mathbf{x}}_\mathbf{i}\}_{i=1}^{d}$.

Step 5. Set $\widehat{X}_N(\eta) = \{\hat{\mathbf{x}}_i\}_{i=1}^{d} \otimes (\mathbf{V}'\mathbf{W}')$.

By the mentioned lemma and relations (4.6) and (4.7), the matrix–vector multiplication representation of the product $\widehat{K}_N(\tau_m, \eta)\widehat{X}_N(\eta)$ can be obtained as follows:

$$\begin{aligned}
\widehat{K}_N(\tau_m, \eta)\widehat{X}_N(\eta) &= \left(\sum_{j=1}^{d} \hat{k}_{ij}(\tau_m, \eta)\hat{x}_j(\eta)\right)_{i=1}^{d} \\
&= \left(\sum_{j=1}^{d} \big((\hat{\mathbf{k}}_\mathbf{ij})_\mathbf{m}\mathbf{V}\mathbf{W}'\big)\big(\hat{\mathbf{x}}_\mathbf{j}\mathbf{V}'\mathbf{W}'\big)\right)_{i=1}^{d} \\
&= \left(\sum_{j=1}^{d}\left(\sum_{l=0}^{N}\big((\hat{\mathbf{k}}_\mathbf{ij})_\mathbf{m}\mathbf{V}\big)_l \eta^l\right)\left(\sum_{l=0}^{N}\big(\hat{\mathbf{x}}_\mathbf{j}\mathbf{V}'\big)_l \eta^l\right)\right)_{i=1}^{d} \\
&= \left(\sum_{j=1}^{d} (\hat{\mathbf{k}}_\mathbf{ij})_\mathbf{m}\mathbf{V}\big(\big(\hat{\mathbf{x}}_\mathbf{j}\mathbf{V}'\big) \otimes \mathbf{M}\big)\mathbf{W}\right)_{i=1}^{d},
\end{aligned} \tag{4.10}$$

where $\mathbf{M} = (\mathbf{M}_N^{(l)})_{l=0}^{N}$ is a block matrix which is defined in Lemma 1.

Inserting (4.10) in (4.9), we obtain

$$\begin{aligned}
\widehat{A}(\tau_m)\{\hat{\mathbf{x}}_\mathbf{i}\}_{i=1}^{d} &= \widehat{G}(\tau_m) + \left(\sum_{j=1}^{d} (\hat{\mathbf{k}}_\mathbf{ij})_\mathbf{m}\mathbf{V}\big(\big(\hat{\mathbf{x}}_\mathbf{j}V'\big) \otimes \mathbf{M}\big)\int_{-1}^{\tau_m} \mathbf{W}\, d\eta\right)_{i=1}^{d} \\
&= \widehat{G}(\tau_m) + \left(\sum_{j=1}^{d} (\hat{\mathbf{k}}_\mathbf{ij})_\mathbf{m}\mathbf{V}\big(\big(\hat{\mathbf{x}}_\mathbf{j}\mathbf{V}'\big) \otimes \mathbf{M}\big)\mathbf{W_m}\right)_{i=1}^{d} \quad (m = 0, 1, \dots, N)
\end{aligned} \tag{4.11}$$

where $\mathbf{W_m} = \int_{-1}^{\tau_m} \mathbf{W}\, d\eta$.

Finally, we end up with a system of equations whose solution gives the unknown vectors $\{\hat{\mathbf{x}}_i\}_{i=1}^{d}$.

The Algorithm 1 summarizes the proposed Jacobi collocation method.

5. Convergence analysis

In this section, we will try to provide an error analysis which theoretically justifies the spectral rate of convergence of the proposed method. In order to describe the key ideas without having to resort to complex notation involving Kronecker products of matrices and vectors, we will consider the index-2 IAEs system (4.3) with $d_1 = d_2 = 1$. Our strategy is mainly based on the ideas in [20], together with the following auxiliary lemmas from [6] and [7]:

Lemma 2. *(See [6].) Let $\phi \in P_N$ where P_N denote the space of all polynomials of degree $\leqslant N$, then for any integer $r \geqslant 1$ and $0 \leqslant p \leqslant \infty$, there exists a positive constant C independent of N such that*

$$\|\phi^{(r)}\|_{L^p_{w^{\alpha,\beta}}(-1,1)} \leqslant CN^{2r}\|\phi\|_{L^p_{w^{\alpha,\beta}}(-1,1)}.$$

Lemma 3. *(See [6].) Assume that* $u \in H^m(\Lambda)$ *and* $P_N u = \sum_{k=0}^{N} \hat{u}_k J_k^{\alpha,\beta}$ *is the truncated orthogonal Jacobi series of* u *and* $I_N^{\alpha,\beta} u$ *is the interpolation of* u *at any of the three families of Jacobi Gauss points (Gauss or Gauss–Radau or Gauss–Lobatto). Then for* $\Lambda = (-1,1)$, *the following estimates hold*:

$$\|u - P_N u\|_{L^2_{w^{\alpha,\beta}}(\Lambda)} \leqslant C N^{-m} |u|_{H^{m,N}(\Lambda)}, \tag{5.1}$$

$$\left\|u - I_N^{\alpha,\beta} u\right\|_{L^2_{w^{\alpha,\beta}}(\Lambda)} \leqslant C N^{-m} |u|_{H^{m,N}_{w^{\alpha,\beta}}(\Lambda)},$$

$$\left\|u' - \left(I_N^{\alpha,\beta} u\right)'\right\|_{L^2_{w^{\alpha,\beta}}(\Lambda)} \leqslant C N^{1-m} |u|_{H^{m,N}_{w^{\alpha,\beta}}(\Lambda)}. \tag{5.2}$$

We are now ready to prove the following main theorem which gives the convergence properties of the presented scheme:

Theorem 3. *Consider the IAEs of index-2 (4.3) and its transformed system (4.5) with* $\widehat{A}(\tau) = \operatorname{diag}(1,0)$, $\widehat{G}(\tau) = (\hat{f}(\tau), \hat{g}(\tau))^T$ *and* $\widehat{X}(t) = (\hat{y}, \hat{z})^T$. *Assume that the hypotheses given in Theorem 1 hold with* $\nu \geqslant 0$. *If* $\widehat{X}_N = (\hat{y}_N, \hat{z}_N)^T$ *is an approximate solution of the proposed Jacobi collocation scheme with the Gauss–Jacobi collocation points, then the Jacobi collocation approximation errors* $\hat{y} - \hat{y}_N$ *and* $\hat{z} - \hat{z}_N$ *for* $m \geqslant 1$ *satisfy*:

$$\|\hat{y} - \hat{y}_N\|_{L^2_{w^{\alpha,\beta}}(\Lambda)} \simeq \begin{cases} \mathcal{O}(N^{2-m} \log N), & -1 < \alpha, \beta \leqslant -\frac{1}{2}, \\ \mathcal{O}(N^{\frac{5}{2}+\gamma-m}), & \textit{otherwise} \end{cases}$$

$$\|\hat{z} - \hat{z}_N\|_{L^2_{w^{\alpha,\beta}}(\Lambda)} \simeq \begin{cases} \mathcal{O}(N^{4-2m} (\log N)^2), & -1 < \alpha, \beta \leqslant -\frac{1}{2}, \\ \mathcal{O}(N^{5+2\gamma-2m}), & \textit{otherwise} \end{cases}$$

where $\hat{y}, \hat{z} \in H^m(\Lambda)$, $\Lambda = (-1,1)$ *and* $\gamma = \max(\alpha, \beta)$.

Proof. The strategy we shall follow is to determine the error estimations for $\|\hat{y} - \hat{y}_N\|$ and then $\|\hat{z} - \hat{z}_N\|$. The spectral collocation solution $\widehat{X}_N$ was considered as an interpolating polynomial of degree N on the interval $(-1,1)$. It is defined through the collocation equation (4.11)

$$\hat{y}(\tau_i) = \hat{f}(\tau_i) + D_1(\tau_i) + D_2(\tau_i) \quad (i = 1, 2, \ldots, N) \tag{5.3}$$

where

$$D_1(\tau_i) = (\hat{\mathbf{k}}_{\mathbf{11}})_{\mathbf{m}} \mathbf{V}\big(\big(\hat{y}\mathbf{V}'\big) \otimes \mathbf{M}\big) \mathbf{W}_{\mathbf{m}}, \qquad D_2(\tau_i) = (\hat{\mathbf{k}}_{\mathbf{12}})_{\mathbf{m}} \mathbf{V}\big(\big(\hat{z}\mathbf{V}'\big) \otimes \mathbf{M}\big) \mathbf{W}_{\mathbf{m}}.$$

We rewrite Eq. (5.3) in the form:

$$\hat{y}(\tau_i) = \hat{f}(\tau_i) + \int_{-1}^{\tau_i} \widehat{K}_{11}(\tau_i, \eta) \hat{y}_N(\eta)\, d\eta + \int_{-1}^{\tau_i} \widehat{K}_{12}(\tau_i, \eta) \hat{z}_N(\eta)\, d\eta + q_1(\tau_i) + q_2(\tau_i), \tag{5.4}$$

such that

$$q_1(\tau_i) = D_1(\tau_i) - \int_{-1}^{\tau_i} \widehat{K}_{11}(\tau_i, \eta) \hat{y}_N(\eta)\, d\eta,$$

$$q_2(\tau_i) = D_2(\tau_i) - \int_{-1}^{\tau_i} \widehat{K}_{12}(\tau_i, \eta) \hat{z}_N(\eta)\, d\eta,$$

and $\hat{y}_N, \hat{z}_N$ are the approximate solutions of $\hat{y}, \hat{z}$ which is approximated by k-th Lagrange interpolating polynomials L_k:

$$I_N^{\alpha,\beta}\big(\hat{y}(\tau)\big) = \hat{y}_N(\tau) = \sum_{k=0}^{n} \hat{y}_k L_k(\tau), \qquad I_N^{\alpha,\beta}\big(\hat{z}(\tau)\big) = \hat{z}_N(\tau) = \sum_{k=0}^{n} \hat{z}_k L_k(\tau),$$

where $\hat{y}_k = \hat{y}(\tau_k)$ and $\hat{z}_k = \hat{z}(\tau_k)$. It follows that

$$\hat{y}(\tau_i)=\hat{f}(\tau_i)+\int_{-1}^{\tau_i}\widehat{K}_{11}(\tau_i,\eta)e(\eta)\,d\eta+\int_{-1}^{\tau_i}\widehat{K}_{11}(\tau_i,\eta)\hat{y}(\eta)\,d\eta$$
$$+\int_{-1}^{\tau_i}\widehat{K}_{12}(\tau_i,\eta)\varepsilon(\eta)\,d\eta+\int_{-1}^{\tau_i}\widehat{K}_{12}(\tau_i,\eta)\hat{z}(\eta)\,d\eta+q_1(\tau_i)+q_2(\tau_i), \tag{5.5}$$

where $e(s)=\hat{y}_N(s)-\hat{y}(s)$ and $\varepsilon(s)=\hat{z}_N(s)-\hat{z}(s)$. Multiplying the j-th equation of (5.5) by $L_j(\tau)$ and summing up over j from 0 to N, we get

$$\hat{y}_N(\tau)=I_N^{\alpha,\beta}\big(\hat{f}(\tau)\big)+I_N^{\alpha,\beta}\left(\int_{-1}^{\tau}\widehat{K}_{11}(\tau,\eta)\hat{y}(\eta)\,d\eta\right)$$
$$+I_N^{\alpha,\beta}\big(q_1(\tau)\big)+I_N^{\alpha,\beta}\big(q_2(\tau)\big)+I_N^{\alpha,\beta}\left(\int_{-1}^{\tau}\widehat{K}_{11}(\tau,\eta)e(\eta)\,d\eta\right)$$
$$+I_N^{\alpha,\beta}\left(\int_{-1}^{\tau}\widehat{K}_{12}(\tau,\eta)\hat{z}(\eta)\,d\eta\right)+I_N^{\alpha,\beta}\left(\int_{-1}^{\tau}\widehat{K}_{12}(\tau,\eta)\varepsilon(\eta)\,d\eta\right). \tag{5.6}$$

From the first equation of (4.5), we may write

$$I_N^{\alpha,\beta}\left(\int_{-1}^{\tau}\widehat{K}_{11}(\tau,\eta)\hat{y}(\eta)\,d\eta+\int_{-1}^{\tau}\widehat{K}_{12}(\tau,\eta)\hat{z}(\eta)\,d\eta\right)=I_N^{\alpha,\beta}\big(\hat{y}(\tau)-\hat{f}(\tau)\big),$$

and replacing the above relation in (5.6), we get

$$0=\int_{-1}^{\tau}\widehat{K}_{11}(\tau,\eta)e(\eta)\,d\eta+\int_{-1}^{\tau}\widehat{K}_{12}(\tau,\eta)\varepsilon(\eta)\,d\eta+I_N^{\alpha,\beta}\big(q_1(\tau)\big)+I_N^{\alpha,\beta}\big(q_2(\tau)\big)+P_1+P_2, \tag{5.7}$$

where

$$P_1=I_N^{\alpha,\beta}\left(\int_{-1}^{\tau}\widehat{K}_{11}(\tau,\eta)e(\eta)\,d\eta\right)-\int_{-1}^{\tau}\widehat{K}_{11}(\tau,\eta)e(\eta)\,d\eta,$$
$$P_2=I_N^{\alpha,\beta}\left(\int_{-1}^{\tau}\widehat{K}_{12}(\tau,\eta)\varepsilon(\eta)\,d\eta\right)-\int_{-1}^{\tau}\widehat{K}_{12}(\tau,\eta)\varepsilon(\eta)\,d\eta.$$

Differentiating from (5.7) respect to τ, yields:

$$-\widehat{K}_{11}(\tau,\tau)e(\tau)-\widehat{K}_{12}(\tau,\tau)\varepsilon(\tau)=\int_{-1}^{\tau}\frac{\partial\widehat{K}_{11}(\tau,\eta)}{\partial\tau}e(\eta)\,d\eta+\int_{-1}^{\tau}\frac{\partial\widehat{K}_{12}(\tau,\eta)}{\partial\tau}\varepsilon(\eta)\,d\eta$$
$$+I_N'^{\alpha,\beta}\big(q_1(\tau)\big)+I_N'^{\alpha,\beta}\big(q_2(\tau)\big)+P_1'+P_2'. \tag{5.8}$$

In order to investigate the convergence of the second equation of (4.5), we apply the vector–matrix multiplication representation form (4.11) for this equation. So, we have

$$0=\hat{g}(\tau_i)+D_3(\tau_i)\quad(i=1,2,\dots,N) \tag{5.9}$$

where $D_3(\tau_i)$ can be written as:

$$D_3(\tau_i)=(\hat{\mathbf{k}}_{\mathbf{21}})_{\mathbf{m}}\mathbf{V}\big(\big(\hat{y}\mathbf{V}'\big)\otimes\mathbf{M}\big)\mathbf{W}_{\mathbf{m}}.$$

Using a similar procedure as outlined in the first part, Eq. (5.9) give rise

$$0=\int_{-1}^{\tau}\widehat{K}_{21}(\tau,\eta)e(\eta)\,d\eta+I_N^{\alpha,\beta}\big(q_3(\tau)\big)+P_3, \tag{5.10}$$

where P_3 and $q_3(\tau_i)$ are similarly defined as:

$$P_3 = I_N^{\alpha,\beta}\left(\int_{-1}^{\tau} \widehat{K}_{21}(\tau,\eta)e(\eta)\,d\eta\right) - \int_{-1}^{\tau} \widehat{K}_{21}(\tau,\eta)e(\eta)\,d\eta,$$

$$q_3(\tau_i) = D_3(\tau_i) - \int_{-1}^{\tau_i} \widehat{K}_{21}(\tau_i,\eta)\hat{y}_N(\eta)\,d\eta.$$

Also, differentiating from (5.10) respect to τ, gives:

$$-\widehat{K}_{21}(\tau,\tau)e(\tau) = \int_{-1}^{\tau} \frac{\partial \widehat{K}_{21}(\tau,\eta)}{\partial \tau} e(\eta)\,d\eta + I_N'^{\alpha,\beta}\big(q_3(\tau)\big) + P_3'. \tag{5.11}$$

Now, let $\mathbf{E}(\tau) = (e(\tau), \varepsilon(\tau))^T$, then Eqs. (5.8) and (5.11), can be rewritten in matrix notation

$$\mathbf{HE}(\tau) = \int_{-1}^{\tau} \widetilde{\mathbf{K}}(\tau,\eta)\mathbf{E}(\eta)\,d\eta + \mathbf{D}, \tag{5.12}$$

with

$$\mathbf{H} = \begin{pmatrix} -\widehat{K}_{11}(\tau,\tau) & -\widehat{K}_{12}(\tau,\tau) \\ -\widehat{K}_{21}(\tau,\tau) & 0 \end{pmatrix}, \qquad \widetilde{\mathbf{K}}(\tau,\eta) = \begin{pmatrix} \frac{\partial \widehat{K}_{11}(\tau,\eta)}{\partial \tau} & \frac{\partial \widehat{K}_{12}(\tau,\eta)}{\partial \tau} \\ \frac{\partial \widehat{K}_{21}(\tau,\eta)}{\partial \tau} & 0 \end{pmatrix},$$

and

$$\mathbf{D} = \begin{pmatrix} I_N'^{\alpha,\beta}(q_1(\tau)) + I_N'^{\alpha,\beta}(q_2(\tau)) + P_1' + P_2' \\ I_N'^{\alpha,\beta}(q_3(\tau)) + P_3' \end{pmatrix}.$$

Due to the assumptions of Theorem 1, we have

$$\big|K_{21}(t,t)K_{12}(t,t)\big| > 0, \quad \forall t \in I,$$

this shows that $\mathbf{H}$ is an invertible matrix with the inverse of the form:

$$\mathbf{H}^{-1} = \begin{pmatrix} 0 & -\widehat{K}_{21}^{-1}(\tau,\eta) \\ -\widehat{K}_{21}^{-1}(\tau,\eta) & \widehat{K}_{12}^{-1}(\tau,\eta)(\widehat{K}_{11}(\tau,\eta)\widehat{K}_{21}^{-1}(\tau,\eta)) \end{pmatrix}.$$

Multiplying (5.12) by $\mathbf{H}^{-1}$ and using the Gronwall's inequality (see e.g. Lemma 3.4 from [20]), we have

$$\|\mathbf{E}\|_{L^2_{w^{\alpha,\beta}}(\Lambda)} \leqslant C\|\mathbf{F}\|_{L^2_{w^{\alpha,\beta}}(\Lambda)}, \tag{5.13}$$

where $\mathbf{F} = \mathbf{H}^{-1}\mathbf{D}$.

Then it follows from Lemma 2:

$$\big\|I_N'^{\alpha,\beta}\big(q_1(\tau)\big)\big\|_{L^2_{w^{\alpha,\beta}}(\Lambda)} \leqslant CN^2\big\|I_N^{\alpha,\beta}\big(q_1(\tau)\big)\big\|_{L^2_{w^{\alpha,\beta}}(\Lambda)}, \tag{5.14}$$

indeed

$$\big\|I_N^{\alpha,\beta}\big(q_1(\tau)\big)\big\|_{L^2_{w^{\alpha,\beta}}(\Lambda)} \leqslant \max_{0\leqslant i\leqslant N}\big|q_1(\tau_i)\big| \max_{\tau\in(\Lambda)} \sum_{i=0}^{n}\big|L_i(\tau)\big|. \tag{5.15}$$

Applying the Cauchy–Schwarz inequality (see e.g. [6]), we can write:

$$\begin{aligned} |q_1(\tau_i)| &= \left|\int_{-1}^{\tau_i} \big((\widehat{K}_{11})_N(\tau_i,\eta) - \widehat{K}_{11}(\tau_i,\eta)\big)\hat{y}_N(\eta)\,d\eta\right| \\ &\leqslant \big\|(\widehat{K}_{11})_N(\tau_i,\eta) - \widehat{K}_{11}(\tau_i,\eta)\big\|_{L^2_{w^{\alpha,\beta}}(\Lambda)}\|\hat{y}_N\|_{L^2_{w^{\alpha,\beta}}(\Lambda)}, \end{aligned}$$

and using (5.1) from Lemma 3, we have

$$\left|q_1(\tau_i)\right| \leqslant CN^{-m}\left|\widehat{K}_{11}(\tau_i,\eta)\right|_{H^{m,N}_{w^{\alpha,\beta}}(\Lambda)}\|\hat{y}_N\|_{L^2_{w^{\alpha,\beta}}(\Lambda)}. \tag{5.16}$$

In this position, we will make use of the result of Chen and Tang [7] and also [18], which gives the Lebesgue constant for the Lagrange interpolating polynomials associated with the nodes of the Jacobi polynomials. Actually, as stated in Lemma 3.4 of [7], the following relation for (5.15) holds

$$\max_{\eta\in[-1,1]}\sum_{i=0}^{n}\left|L_i(\eta)\right| = \begin{cases}\mathcal{O}(\log N), & -1<\alpha,\beta\leqslant -\frac{1}{2},\\ \mathcal{O}(N^{\gamma+\frac{1}{2}}), \quad \gamma=\max(\alpha,\beta), & \text{otherwise}\end{cases}$$

so, we have

$$\left\|I_N^{\alpha,\beta}\big(q_1(\tau)\big)\right\|_{L^2_{w^{\alpha,\beta}}(\Lambda)} \leqslant \begin{cases} CN^{-m}\log N\Theta_{11}, & -1<\alpha,\beta\leqslant-\frac{1}{2},\\ CN^{\frac{1}{2}+\gamma-m}\Theta_{11}, & \text{otherwise}\end{cases}$$

where $\Theta_{11}=\max_{0\leqslant i\leqslant N}|\widehat{K}_{11}(\tau_i,\eta)|_{H^{m,N}_{w^{\alpha,\beta}}(\Lambda)}\|\hat{y}_N\|_{L^2_{w^{\alpha,\beta}}(\Lambda)}$ and $\gamma=\max(\alpha,\beta)$.

Now, by using this relation for (5.14), we have

$$\left\|I_N'^{\alpha,\beta}\big(q_1(\tau)\big)\right\|_{L^2_{w^{\alpha,\beta}}(\Lambda)} \leqslant \begin{cases} CN^{2-m}\log N\Theta_{11}, & -1<\alpha,\beta\leqslant-\frac{1}{2},\\ CN^{\frac{5}{2}+\gamma-m}\Theta_{11}, & \text{otherwise}\end{cases}$$

and similarly

$$\left\|I_N'^{\alpha,\beta}\big(q_2(\tau)\big)\right\|_{L^2_{w^{\alpha,\beta}}(\Lambda)} \leqslant \begin{cases} CN^{2-m}\log N\Theta_{12}, & -1<\alpha,\beta\leqslant-\frac{1}{2},\\ CN^{\frac{5}{2}+\gamma-m}\Theta_{12}, & \text{otherwise}\end{cases}$$

$$\left\|I_N'^{\alpha,\beta}\big(q_3(\tau)\big)\right\|_{L^2_{w^{\alpha,\beta}}(\Lambda)} \leqslant \begin{cases} CN^{2-m}\log N\Theta_{21}, & -1<\alpha,\beta\leqslant-\frac{1}{2},\\ CN^{\frac{5}{2}+\gamma-m}\Theta_{21}, & \text{otherwise}\end{cases}$$

where

$$\Theta_{12}=\max_{0\leqslant i\leqslant N}\left|\widehat{K}_{12}(\tau_i,\eta)\right|_{H^{m,N}_{w^{\alpha,\beta}}(\Lambda)}\|\hat{z}_N\|_{L^2_{w^{\alpha,\beta}}(\Lambda)},$$
$$\Theta_{21}=\max_{0\leqslant i\leqslant N}\left|\widehat{K}_{21}(\tau_i,\eta)\right|_{H^{m,N}_{w^{\alpha,\beta}}(\Lambda)}\|\hat{y}_N\|_{L^2_{w^{\alpha,\beta}}(\Lambda)}.$$

Indeed, using (5.2) and (4.1) for $m=1$ and then applying Hardy's inequality (see e.g. Lemma 3.7 from [7]), we can write

$$\begin{aligned}
\left\|P_1'\right\|_{L^2_{w^{\alpha,\beta}}(\Lambda)} &\leqslant C\left\|\widehat{K}_{11}(\tau,\tau)e(\tau)+\int_{-1}^{\tau}\frac{\partial(\widehat{K}_{11}(\tau,\eta))}{\partial\tau}e(\eta)\,d\eta\right\|_{L^2_{w^{\alpha,\beta}}(\Lambda)}\\
&\leqslant C\left\{\left\|\widehat{K}_{11}(\tau,\tau)\right\|_{L^2_{w^{\alpha,\beta}}(\Lambda)}\left\|e(\tau)\right\|_{L^2_{w^{\alpha,\beta}}(\Lambda)}+\left\|\int_{-1}^{\tau}\frac{\partial(\widehat{K}_{11}(\tau,\eta))}{\partial\tau}e(\eta)\,d\eta\right\|_{L^2_{w^{\alpha,\beta}}(\Lambda)}\right\}\\
&\leqslant C\left\{\left\|\widehat{K}_{11}(\tau,\tau)\right\|_{L^2_{w^{\alpha,\beta}}(\Lambda)}\left(CN^{-m}|\hat{y}|_{H^{m,N}_{w^{\alpha,\beta}}(\Lambda)}\right)+C\left\|e(\tau)\right\|_{L^2_{w^{\alpha,\beta}}(\Lambda)}\right\}\\
&\leqslant C\left\{\left(\left\|\widehat{K}_{11}(\tau,\tau)\right\|_{L^2_{w^{\alpha,\beta}}(\Lambda)}+C\right)\left(CN^{-m}|\hat{y}|_{H^{m,N}_{w^{\alpha,\beta}}(\Lambda)}\right)\right\},
\end{aligned}$$

thus

$$\left\|P_1'\right\|_{L^2_{w^{\alpha,\beta}}(\Lambda)} \leqslant CN^{-m}\left(\left\|\widehat{K}_{11}(\tau,\tau)\right\|_{L^2_{w^{\alpha,\beta}}(\Lambda)}+C\right)|\hat{y}|_{H^{m,N}_{w^{\alpha,\beta}}(\Lambda)} \leqslant CN^{-m}|\hat{y}|_{H^{m,N}_{w^{\alpha,\beta}}(\Lambda)}.$$

Similarly

$$\left\|P_2'\right\|_{L^2_{w^{\alpha,\beta}}(\Lambda)} \leqslant CN^{-m}\left(\left\|\widehat{K}_{12}(\tau,\tau)\right\|_{L^2_{w^{\alpha,\beta}}(\Lambda)}+C\right)|\hat{z}|_{H^{m,N}_{w^{\alpha,\beta}}(\Lambda)} \leqslant CN^{-m}|\hat{z}|_{H^{m,N}_{w^{\alpha,\beta}}(\Lambda)},$$
$$\left\|P_3'\right\|_{L^2_{w^{\alpha,\beta}}(\Lambda)} \leqslant CN^{-m}\left(\left\|\widehat{K}_{21}(\tau,\tau)\right\|_{L^2_{w^{\alpha,\beta}}(\Lambda)}+C\right)|\hat{y}|_{H^{m,N}_{w^{\alpha,\beta}}(\Lambda)} \leqslant CN^{-m}|\hat{y}|_{H^{m,N}_{w^{\alpha,\beta}}(\Lambda)}.$$

The above estimates together with (5.13), yield

$$\|e\|_{L^2_{w^{\alpha,\beta}}(\Lambda)} = \|\hat{y}-\hat{y}_N\|_{L^2_{w^{\alpha,\beta}}(\Lambda)} \leqslant CN^{-m}|\hat{y}|_{H^{m,N}_{w^{\alpha,\beta}}(\Lambda)} + \begin{cases} CN^{2-m}\log N\Theta_{21}, & -1<\alpha,\beta\leqslant-\frac{1}{2},\\ CN^{\frac{5}{2}+\gamma-m}\Theta_{21}, & \text{otherwise}\end{cases} \tag{5.17}$$

Table 1
$L^2_{w^{\alpha,\beta}}$ errors for Example 1.

N	$\|x_1-u\|_{L^2_{w^{\alpha,\beta}}}$	$\|x_2-v\|_{L^2_{w^{\alpha,\beta}}}$	$\|x_3-w\|_{L^2_{w^{\alpha,\beta}}}$
4	1.79×10^{-4}	9.86×10^{-5}	3.72×10^{-3}
6	3.12×10^{-6}	1.84×10^{-6}	1.30×10^{-4}
8	7.66×10^{-8}	4.67×10^{-8}	4.61×10^{-6}
10	1.99×10^{-9}	1.46×10^{-9}	1.59×10^{-7}
12	5.33×10^{-11}	3.01×10^{-11}	5.38×10^{-9}
14	1.37×10^{-12}	7.73×10^{-13}	1.74×10^{-10}

Also Θ_{11} can be written as:

$$\Theta_{11}=\max_{0\leqslant i\leqslant N}|\widehat{K}_{11}(\tau_i,\eta)|_{H^{m,N}_{w^{\alpha,\beta}}(\Lambda)}\|\hat{y}_N\|_{L^2_{w^{\alpha,\beta}}(\Lambda)}\leqslant K(\|\hat{y}\|_{L^2_{\omega}(\Lambda)}+\|e\|_{L^2_{\omega}(\Lambda)}),$$

where $K=\max_{0\leqslant i\leqslant N}|\widehat{K}_{11}(\tau_i,\eta)|_{H^{m,N}_{w^{\alpha,\beta}}(\Lambda)}$.

Finally, the above estimates together with (5.13) and (5.17) give

$$\begin{aligned}\|\varepsilon\|_{L^2_{w^{\alpha,\beta}}(\Lambda)}&=\|\hat{z}-\hat{z}_N\|_{L^2_{w^{\alpha,\beta}}(\Lambda)}\leqslant CN^{-m}\big(|\hat{y}|_{H^{m,N}_{w^{\alpha,\beta}}(\Lambda)}+|\hat{z}|_{H^{m,N}_{w^{\alpha,\beta}}(\Lambda)}\big)\\&\quad+\begin{cases}CN^{2-m}\log N(K(\|\hat{y}\|_{L^2_{w^{\alpha,\beta}}(\Lambda)}+\|e\|_{L^2_{w^{\alpha,\beta}}(\Lambda)})+\Theta_{12}+\Theta_{21}), & -1<\alpha,\beta\leqslant-\frac{1}{2},\\ CN^{\frac{5}{2}+\gamma-m}(K(\|\hat{y}\|_{L^2_{w^{\alpha,\beta}}(\Lambda)}+\|e\|_{L^2_{w^{\alpha,\beta}}(\Lambda)})+\Theta_{12}+\Theta_{21}), & \text{otherwise}\end{cases}\end{aligned}$$

which leads to the estimate stated of the theorem, provided that N is sufficiently large and C is a constant independent of N. □

6. Numerical results and discussions

To incorporate our numerical approach, two semi-explicit IAEs system of index-2 together an applied problem are considered. These problems are solved using the proposed Jacobi collocation method for $\alpha=\frac{1}{2}$ and $\beta=\frac{1}{3}$ based on matrix–vector multiplication representation of equations. To examine the accuracy of the results, $L^2_{w^{\alpha,\beta}}$ errors are employed to assess the efficiency of the method. All the calculations were supported by the software Matematica®.

Example 1. Consider the following semi-explicit linear IAEs system of index-2

$$A(t)X(t)=g(t)+\int_0^t K(t,s)X(s)\,ds,$$

where

$$A(t)=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix},\qquad K(t,s)=\begin{pmatrix}\frac{3-2s}{2-s}&\frac{3-s}{2-s}&2(2-s)\\ \frac{-1}{s-2}&-1&1\\ s+2&s^2-4&0\end{pmatrix},$$

$$X(t)=\big(x_1(t),x_2(t),x_3(t)\big)^T,\qquad g(t)=\big(1,2e^t-1,-1+e^t\big(-t^2+t+1\big)\big)^T.$$

The exact solutions of the system are:

$$x_1(t)=x_2(t)=e^t,\qquad x_3(t)=-\frac{e^t}{2-t}.$$

Let u,v,w be the approximation of the exact solutions x_1,x_2,x_3, respectively, that is given by (4.7). The discrete method described in Section 4 has been implemented for the problem and the $L^2_{w^{\alpha,\beta}}$ errors for different values of $N=4,6,8,\ldots$ have been reported in Table 1. Graphs of the error functions and error behaviors for several values of N are also given in Figs. 1 and 2. We observe that the approximate solution of the equation represents the expected convergence behavior as described in Theorem 3.

Example 2. Let us consider the IAEs system of index-2:

$$A(t)X(t)=g(t)+\int_0^t K(t,s)X(s)\,ds,$$

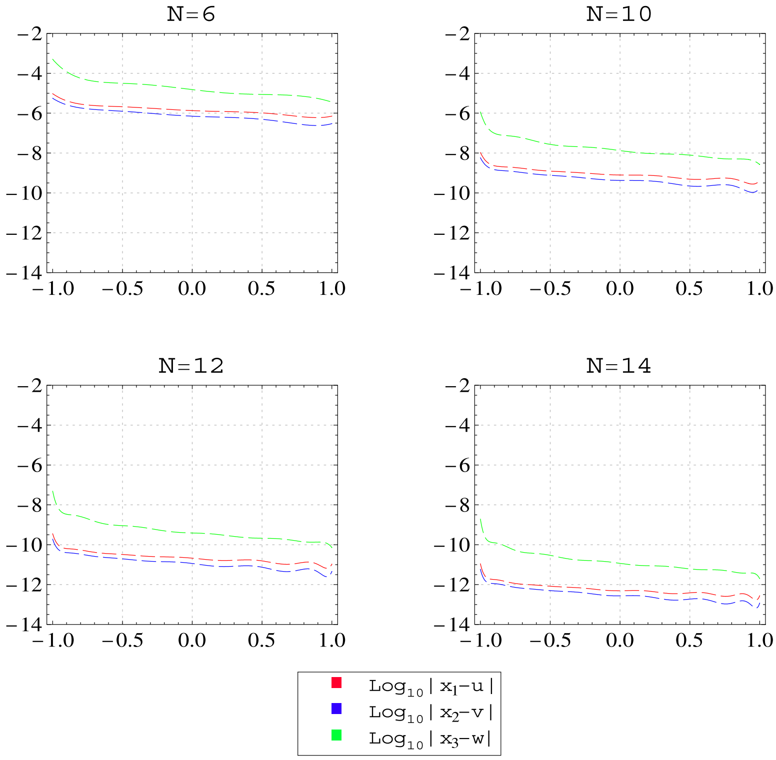

Fig. 1. Error functions of the Jacobi collocation approximations of orders $N = 6, 10, 12$ and 14 for $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{3}$ in Example 1.

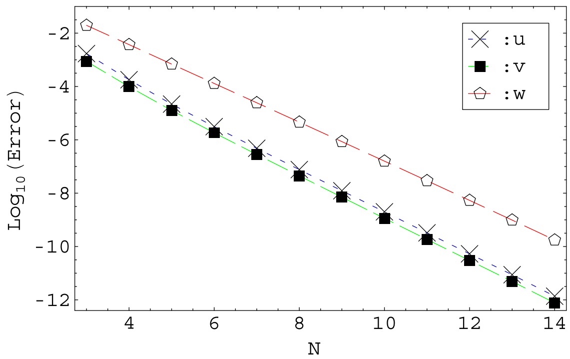

Fig. 2. $L^2_{w^{\alpha,\beta}}$ errors versus the number of collocation points for Example 1.

with

$$A(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad K(t,s) = \begin{pmatrix} e^{t+s} & (s+1)^2 \\ s+t+2 & 0 \end{pmatrix},$$
$$X(t) = \big(x_1(t), x_2(t)\big)^T, \qquad g(t) = \big(f_1(t), f_2(t)\big)^T,$$

and

$$f_1(t) = \sin t - \frac{1}{2}e^t\big(1 + e^t(-\cos t + \sin t)\big) - \frac{1}{4}\big(-2 + 2(1+t)\cos 2t + \big(1 + 4t + 2t^2\big)\sin 2t\big),$$
$$f_2(t) = -(2+t) + 2(1+t)\cos t - \sin t.$$

The exact solutions of the system are:

$$x_1(t) = \sin t, \qquad x_2(t) = \cos 2t.$$

We assume u and v are the Jacobi collocation approximation of the exact solutions x_1 and x_2 respectively, which are defined by (4.7).

Table 2
$L^2_{w^{\alpha,\beta}}$ errors for Example 2.

N	$\|x_1 - u\|_{L^2_{w^{\alpha,\beta}}}$	$\|x_2 - v\|_{L^2_{w^{\alpha,\beta}}}$
4	2.41×10^{-5}	5.29×10^{-4}
6	4.25×10^{-8}	4.44×10^{-6}
8	4.18×10^{-11}	1.86×10^{-8}
10	2.39×10^{-14}	4.85×10^{-11}
12	4.05×10^{-15}	1.97×10^{-13}
14	3.94×10^{-15}	1.45×10^{-13}

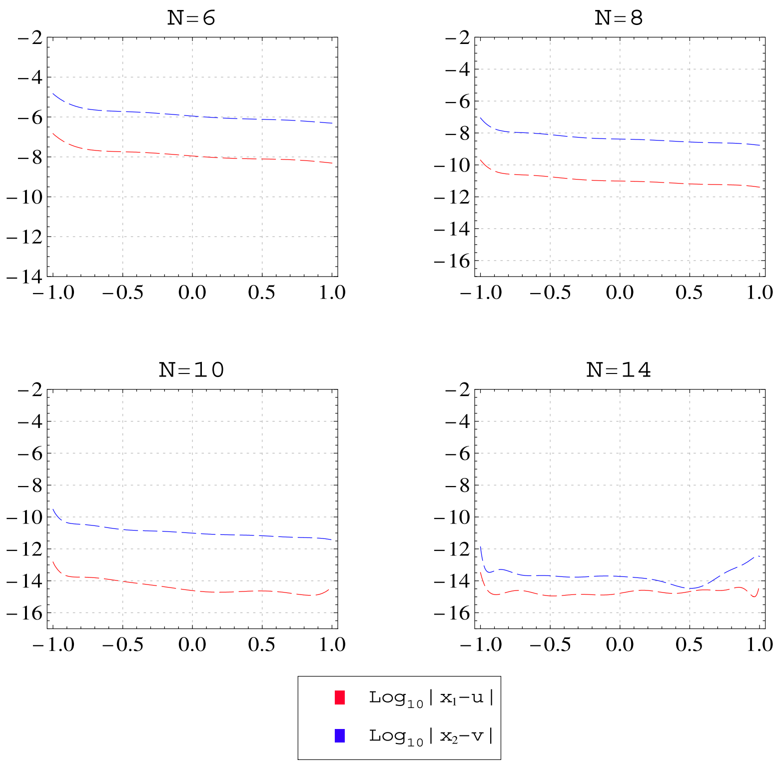

Fig. 3. Error functions of the Jacobi collocation approximations of orders $N = 6, 8, 10$ and 14 for $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{3}$ in Example 2.

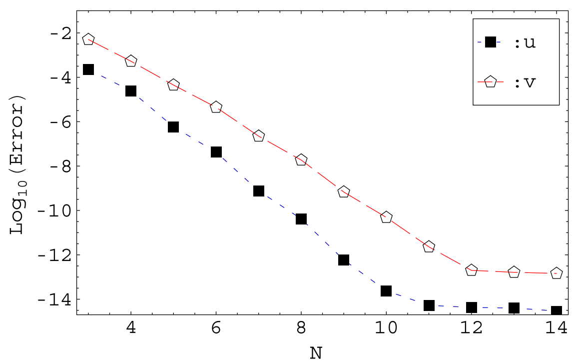

Fig. 4. $L^2_{w^{\alpha,\beta}}$ errors versus the number of collocation points for Example 2.

The computational results have been reported in Table 2. Figs. 3 and 4 show the graphs of the Jacobi collocation error functions with $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{3}$. It is observed that the errors decay exponentially.

Table 3
The absolute errors of Example 3 for different values of N.

N	$\lvert u(0.95, 0.05) - \hat{u}(0.95, 0.05)\rvert$	$\lvert u(1, 0.05) - \hat{u}(1, 0.05)\rvert$
2	1.69×10^{-3}	5.53×10^{-6}
3	1.59×10^{-3}	3.47×10^{-6}
4	1.49×10^{-3}	2.54×10^{-6}
5	1.48×10^{-3}	1.98×10^{-6}

Example 3. As an applied test problem, consider the controlled heat equation (1.2) with $f(x) = 1$, $\alpha(t) = 1$, $\beta(t) = 0$, $h(t) = \frac{1}{\sqrt{\pi t}}$, and $g(t) = \frac{e^{\frac{-1}{4t}}(1+2t)}{2\sqrt{\pi} t^{\frac{3}{2}}}$.

The exact solution of the equation is:

$$u(x,t) = \operatorname{erf}\left(\frac{1-x}{2\sqrt{t}}\right).$$

It can be shown that, under the assumption $\gamma(t) = \frac{e^{\frac{-49}{4t}}(7-12e^{\frac{13}{4t}}+5e^{\frac{6}{t}})(9+\pi^2 t)}{9\pi t(-\operatorname{erf}(\frac{5}{2\sqrt{t}})+2\operatorname{erf}(\frac{3}{\sqrt{t}})-\operatorname{erf}(\frac{7}{2\sqrt{t}}))}$, the system (1.4) can be reduced to a system of the form (1.1) as follows

$$\begin{cases} \dfrac{e^{\frac{-1}{4t}}(1+2t)}{2\sqrt{\pi} t^{\frac{3}{2}}} = \phi_1(t) - 2\displaystyle\int_0^t \theta(0, t-\tau)\phi_1(\tau)\,d\tau + 2\int_0^t \theta(-1, t-\tau)\phi_2(\tau)\,d\tau, \\ \dfrac{1}{\sqrt{\pi t}} = -2\displaystyle\int_0^t \theta(1, t-\tau)\phi_1(\tau)\,d\tau, \end{cases}$$

where $\theta(0, t-\tau) \simeq \frac{1}{\sqrt{4\pi t}}(1 + \frac{\pi^2}{3}t)$, $\theta(-1, t-\tau) \simeq \frac{3}{4}\sqrt{t}\pi^{\frac{3}{2}}$ and $\theta(1, t-\tau) \simeq \frac{1}{2}\sqrt{t}\pi^{\frac{3}{2}}$.

Let $\hat{\phi}_1(t)$ and $\hat{\phi}_2(t)$ be the approximation of the exact solutions $\phi_1(t)$ and $\phi_2(t)$ that are given by (4.7). Now, by replacing $\hat{\phi}_1(t)$ and $\hat{\phi}_2(t)$ in (1.3), the approximate solution of the problem (1.2) can be obtained as

$$\hat{u}(x,t) = \int_0^1 \left\{\hat{\theta}(x-\xi, t) - \hat{\theta}(x+\xi, t)\right\} d\xi - 2\int_0^t \frac{\partial\hat{\theta}}{\partial x}(x, t-\tau)\left\{1 + \int_0^\tau \hat{\phi}_1(\eta)\,d\eta\right\} d\tau$$
$$+ 2\int_0^t \frac{\partial\hat{\theta}}{\partial x}(x-1, t-\tau)\left\{1 + \int_0^\tau \hat{\phi}_2(\eta)\,d\eta\right\} d\tau, \tag{6.1}$$

where $\hat{\theta}(x,t)$ is the truncated series of the Theta function

$$\hat{\theta}(x,t) = \frac{1}{\sqrt{4\pi t}} \sum_{m=-3}^{3} \exp\left(-\frac{(x+2m)^2}{4t}\right).$$

The approximate solution of $\hat{u}(0.95, 0.05)$ and $\hat{u}(1, 0.05)$ are special interest and computed using the Euler product integration method for (6.1). The differences between the numerical and analytical solutions for different values of N, are listed in Table 3.

7. Conclusion

Here, we elaborated a spectral collocation method based on Jacobi orthogonal polynomials to obtain numerical solution of a class of IAEs system with index-2. We will achieve the goal by using the matrix–vector representation and the solution of the related system of equations. The strategy has been derived using some variable transformations to change the equation into an other IAEs defined on the standard interval $[-1, 1]$, so the Jacobi orthogonal polynomial theory can be applied conveniently and the obtained index-2 system is solved directly without using the index reduction procedure which causes the simplification of the method. The spectral rate of convergence for the proposed method established in weighted L^2 norm.

With the availability of this methodology, it will now be possible to investigate the approximate solution of other classes of IAEs systems. Although our convergence theory does not cover the nonlinear case, but it contains some complications and restrictions for establishing a convergent result similar to Theorem 3 which will be the subject of our future work.

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References

[1] U. Ascher, L.R. Petzold, Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations, SIAM, 1998.
[2] H. Brunner, Collocation Methods for Volterra Integral and Related Functional Equations, University Press, Cambridge, 2004.
[3] M.V. Bulatov, V.F. Chistyakov, The properties of differential-algebraic systems and their integral analogs, Memorial University of Newfoundland, preprint, 1997.
[4] M.V. Bulatov, Numerical methods of investigation and solution for integral algebraic equations, in: SciCADE09, May 25–29, 2009.
[5] J.R. Cannon, The One-Dimensional Heat Equation, University Press, Cambridge, 1984.
[6] C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zang, Spectral Methods Fundamentals in Single Domains, Springer-Verlag, 2006.
[7] Y. Chen, T. Tang, Convergence analysis of the Jacobi spectral collocation methods for Volterra integral equations with a weakly singular kernel, Math. Comput. 79 (2010) 147–167.
[8] C.W. Gear, Differential-algebraic equations, indices, and integral-algebraic equations, SIAM. J. Numer. Anal. 27 (1990) 1527–1534.
[9] C.W. Gear, Differential-algebraic equations and index transformation, SIAM. J. Stat. Comput. 1 (1988) 39–47.
[10] A.M. Gomilko, A Dirichlet problem for the biharmonic equation in a semi-infinite strip, J. Engng. Math. 46 (2003) 253–268.
[11] B. Guo, L. Wang, Jacobi interpolation approximations and their applications to singular differential equations, Adv. Comput. Math. 14 (2001) 227–276.
[12] M. Hanke, E.I. Macana, R. März, On asymptotics in case of linear index-2 differential-algebraic equations, SIAM. J. Numer. Anal. 35 (1998) 1326–1346.
[13] I. Higueras, R. März, C. Tischendorf, Stability preserving integration of index-2 DAEs, Appl. Numer. Math. 45 (2/3) (2003) 201–229.
[14] B. Jumarhon, W. Lamb, S. McKee, T. Tang, A Volterra integral type method for solving a class of nonlinear initial-boundary value problems, Numer. Methods Partial Differential Equations 12 (1996) 265–281.
[15] V.V. Kafarov, B. Mayorga, C. Dallos, Mathematical method for analysis of dynamic processes in chemical reactors, Chem. Engng. Sci. 54 (1999) 4669–4678.
[16] J.P. Kauthen, The numerical solution of integral-algebraic equations of index-1 by polynomial spline collocation methods, Math. Comp. 236 (2000) 1503–1514.
[17] R. März, The index of linear differential-algebraic equations with properly stated leading term, Results Math. 42 (3/4) (2002) 308–338.
[18] C.K. Qu, R. Wong, Szego's conjecture on Lebesgue costants for Legendre series, Pacific. J. Math. 135 (1988) 157–188.
[19] J. Shen, T. Tang, Spectral and High-Order Methods with Applications, Science Press, Beijing, 2006.
[20] T. Tang, X. Xu, J. Cheng, On spectral methods for Volterra type integral equations and the convergence analysis, J. Comput. Math. 26 (2008) 825–837.
[21] L.V. Wolfersdorf, On identification of memory kernel in linear theory of heat conduction, Math. Methods Appl. Sci. 17 (1994) 919–932.
[22] A.I. Zenchuk, Combination of inverse spectral transform method and method of characteristics: deformed Pohlmeyer equation, J. Nonlinear Math. Physics 15 (2008) 437–448.