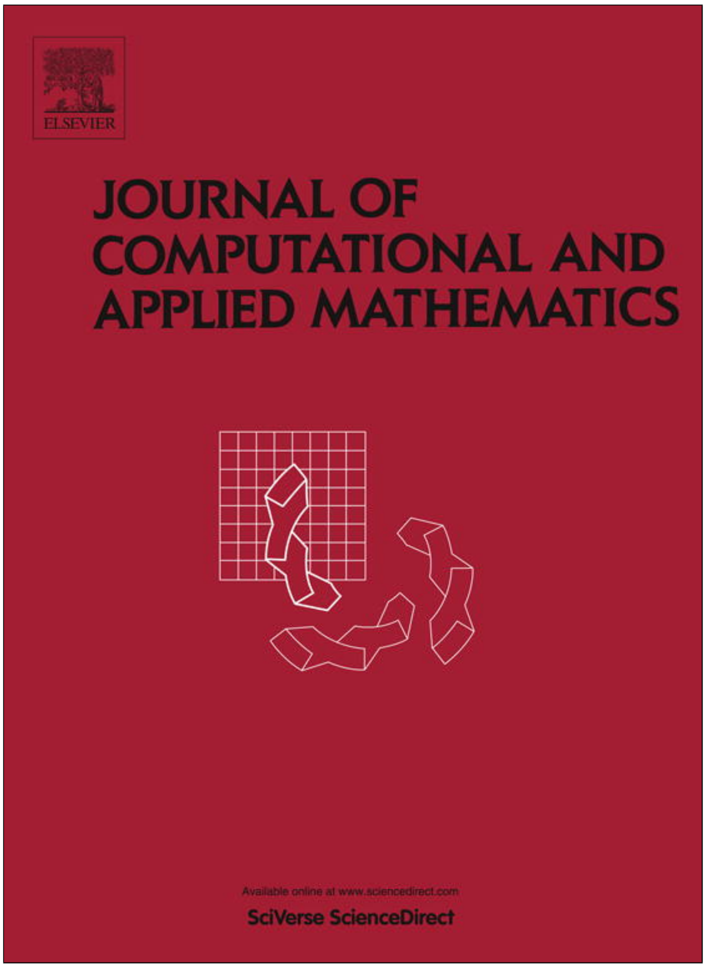

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The semi-explicit Volterra integral algebraic equations with weakly singular kernels: The numerical treatments

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ABSTRACT

This paper deals with some theoretical and numerical results for Volterra Integral Algebraic Equations (*IAEs*) of index-1 with weakly singular kernels. This type of equations typically has solutions whose derivatives are unbounded at the left endpoint of the interval of integration. For overcoming this non-smooth behavior of solutions, using the appropriate coordinate transformation the primary system is changed into a new IAEs which its solutions have better regularity. An effective numerical method based on the Chebyshev collocation scheme is designed and its convergence analysis is provided. Our numerical experiments show that the theoretical results are in good accordance with actual convergence rates obtained by the given algorithm.

1. Introduction

In this paper, our aim is to present a numerical method for the solution of a mixed system of Volterra integral equations of the first and second kind with weakly singular kernels which is known as *Weakly Singular Integral Algebraic Equations* (*WSIAEs*). More precisely, we consider the following system of Volterra integral equations:

$$AX(t) = G(t) + \int_0^t (t-s)^{-\alpha} K(t,s)X(s)ds, \quad 0 < \alpha < 1, \ t \in I = [0,T], \tag{1.1}$$

where $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is a singular matrix, $0 < \alpha = \frac{p}{q} < 1$ ($p, q \in \mathbb{N}, \ p < q$), and

$$K(t,s) = \begin{bmatrix} K_{11}(t,s) & K_{12}(t,s) \\ K_{21}(t,s) & K_{22}(t,s) \end{bmatrix}, \qquad X(t) = (y(t), z(t))^T, \qquad G(t) = (f(t), g(t))^T,$$

such that the functions f, g and the kernels $K_{kl}(\cdot,\cdot)$ ($k, l = 1, 2$) are known smooth functions on I, respectively, and $D = \{(t,s) : 0 \le s \le t \le T\}$. Throughout the paper we assume that the given functions g and K_{22} satisfy in the following relations

$$g(0) = 0, \qquad |K_{22}(t,t)| \ge k_0 > 0, \quad t \in I.$$

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Under these conditions, the system (1.1) is called the weakly singular integral algebraic equations of index-1.

A good source on applications of IAEs with weakly singular kernels is the initial (boundary) value problems for the semi-infinite strip and temperature boundary specification including two/three-phase inverse Stefan problems [1–4], so that these problems consist of a reconstruction of the function describing the coefficient of heat-transfer, when the positions of the moving solid and liquid interfaces are well-known. Also, the monograph of Brunner [5] contains a wide ranging description of IAEs arising in applications from memory kernel identification problems in heat conduction and viscoelasticity [6–8] to the evolution of a chemical reaction within a small cell [9].

There have been a few works available in the literature which investigate the numerical methods for IAEs (see e.g. [5,10–14]). However, as far as we know the numerical analysis of IAEs with weakly singular kernels is largely incomplete and this is a new topic for research. The existence and uniqueness results for the solution of WSIAEs has been given in [15] and most recently in [16]. The piecewise polynomial collocation method for IAEs (1.1) together the concept of tractability index have also been considered by Brunner [5], so that he analyzed the regularity of the solutions using conditions that hold for the first and second kind Volterra integral equations.

Generally, this type of IAEs typically has a solution whose derivatives are unbounded at $t = 0$ and we have to use a suitable strategy to restore this difficulty. Here, we apply an approach like the idea of Li and Tang in [17]. We first consider a suitable function transformation to change the system (1.1) into a new IAE such that its solutions have better regularity. Then, we employ the Chebyshev collocation method to approximate solution of the resulting WSIAE. It is well-known that the methods based on Chebyshev polynomials play a key role in the context of spectral methods. Their widespread use can be traced back to a number of reasons. Not only are the polynomials given in a simple form but all the Gauss quadrature nodes and the associated weights are also given in closed form. Here, for computational purposes, we employ the Chebyshev collocation method to obtaining high accuracy of the results and then a rigorous error analysis is provided in the weighted L^2-norm which shows the spectral rate of convergence is attained. Finally, some numerical examples with the aim of illustrating the convergence behavior of the method are presented.

2. Some basic and auxiliary results

This section is devoted to discussing how the weakly singular IAEs can be changed to treat the problem. Furthermore, the index concept for WSIAEs which plays a fundamental role in both the analysis and the development of numerical algorithms for IAEs is discussed.

2.1. The index for weakly singular IAEs

One of the main features of IAEs systems is their index, which reveals the mathematical structure, potential complications and their numerical solvability. Generally, the difficulties are arising in the theoretical and numerical analysis of IAEs relevant to the index notion. There are several definitions of index in literature not all completely equivalent. For instance, differentiation index [18], the left index [10] and the tractability index (see, e.g. Definition (8.1.7) from [5]).

Here, we use the concept of the *differentiation index* which measures, loosely speaking, how far the main WSIAE is apart from a regular system of VIEs. This notion for WSIAEs discusses by means of the study of the ranks of certain Jacobian associated sub-matrices. In other words, the number of analytical differentiations of the system (1.1) until it can be formulated as a regular system of Volterra integral equations is called differentiation index.

Let us consider the index-1 WSIAE (1.1). Using the classical theory of Volterra integral equations with weakly singular kernels from [10, p. 353], if we multiply both sides of the second equation of (1.1) by the factor $\frac{dt}{(u-t)^{1-\alpha}}$ and integrate with respect to t, the following first kind Volterra integral equation with regular bounded kernels will be obtained

$$0 = \int_0^t H_{21}(t,s)y(s)ds + \int_0^t H_{22}(t,s)z(s)ds + G_\alpha(g), \tag{2.1}$$

where

$$H_{21}(t,s) = \int_0^1 \frac{K_{21}(s+(t-s)\nu, s)}{\nu^\alpha(1-\nu)^{1-\alpha}}d\nu, \qquad H_{22}(t,s) = \int_0^1 \frac{K_{22}(s+(t-s)\nu, s)}{\nu^\alpha(1-\nu)^{1-\alpha}}d\nu,$$

and

$$G_\alpha(g) = \int_0^t (t-s)^{\alpha-1}g(s)ds.$$

Also, differentiation the Eq. (2.1), gives the following second kind integral equation:

$$0 = H_{21}(t,t)y(t) + H_{22}(t,t)z(t) + \int_0^t \frac{\partial H_{21}(t,s)}{\partial t}y(s)ds + \int_0^t \frac{\partial H_{22}(t,s)}{\partial t}z(s)ds + G'_\alpha(g), \tag{2.2}$$

where $H_{2j}(t,t) = \frac{\sin(\alpha\pi)}{\pi} K_{2j}(t,t)$ $(j = 1, 2)$ and $G'_\alpha(g)$ can be obtained using integration by parts to $G_\alpha(g)$ with $g(0) = 0$, as

$$G'_\alpha(g) = \int_0^t (t-s)^{\alpha-1} g'(s)ds.$$

Since $|K_{22}(t,t)| \geq k_0 > 0$, we have $|H_{22}(t,t)| > 0$, then (2.2) together with the first equation of (1.1) is a system of regular Volterra integral equation.

However, it has to be pointed out this reduction (differentiation) is NOT practical from a numerical point of view and such a definition may be useful for understanding the underlying mathematical structure of a WSIAE, and hence choosing an appropriate numerical method for their solutions.

2.2. Smoothness of the solution

Let us assume that the Hölder space $C^{0,\beta}([0,T]) = C^\beta([0,T])$ is defined as a subspace of $C([0,T])$ that consists of functions which are Hölder continuous with the exponent β. More generally, for $k \in \mathbb{Z}_+$ and $\beta \in (0, 1]$, we define the Hölder space

$$C^{k,\beta}([0,T]) = \left\{ f \in C^k([0,T]) \mid D^\nu f \in C^{0,\beta}([0,T])\ \forall \nu,\ |\nu| = k \right\}.$$

In [19], it is shown that this space is a Banach space with the following norm:

$$\|f\|_{C^{k,\beta}([0,T])} = \|f\|_{C^k([0,T])} + \sum_{|\nu|=k} \sup \left\{ \frac{|D^\nu f(x) - D^\nu f(y)|}{|x-y|^\beta} \,\middle|\, x, y \in [0,T],\ x \neq y \right\}.$$

Consider the following system of index-1 WSIAEs:

$$\begin{cases} y(t) = f(t) + \int_0^t (t-s)^{-\alpha} K_{11}(t,s) y(s) ds + \int_0^t (t-s)^{-\alpha} K_{12}(t,s) z(s) ds, \\ 0 = g(t) + \int_0^t (t-s)^{-\alpha} K_{21}(t,s) y(s) ds + \int_0^t (t-s)^{-\alpha} K_{22}(t,s) z(s) ds, \end{cases} \tag{2.3}$$

where $0 < \alpha < 1$ and $t \in I = [0, T]$.

Here, smooth forcing given functions lead to a solution which has typically unbounded derivatives at $t = 0$. The degree of regularity of y and z follows essentially from the corresponding discussions in [5], for the first and second kind weakly singular Volterra integral equations. (See e.g. Theorems 8.1.8, 6.1.6 and 6.1.14 from [10, pp. 346, 354, 478].) In [5], it is shown that the solutions $y(t)$ and $z(t)$ lie in the Hölder spaces $C^{1-\alpha}(I)$ and $C^\alpha(I)$, respectively. This indicates that for any positive integer m, the solutions $y(t)$ and $z(t)$ do not belong to $C^m(I)$. In order to overcome this drawback, we may apply a strategy, like the idea of Li and Tang in [17]. This is done by introducing the following transformations

$$t = u^q, \quad u = \sqrt[q]{t}, \qquad s = w^q, \quad w = \sqrt[q]{s}, \tag{2.4}$$

to change (2.3) to the following system

$$\begin{cases} \hat{y}(u) = \hat{f}(u) + \int_0^u (u-w)^{-\alpha} \hat{K}_{11}(u,w) \hat{y}(w) dw + \int_0^u (u-w)^{-\alpha} \hat{K}_{12}(u,w) \hat{z}(w) dw, \\ 0 = \hat{g}(u) + \int_0^u (u-w)^{-\alpha} \hat{K}_{21}(u,w) \hat{y}(w) dw + \int_0^u (u-w)^{-\alpha} \hat{K}_{22}(u,w) \hat{z}(w) dw, \quad u \in [0, \sqrt[q]{T}], \end{cases} \tag{2.5}$$

where $\hat{f}(u) = f(u^q),\ \hat{g}(u) = g(u^q),\ \hat{y}(u) = y(u^q),\ \hat{z}(u) = z(u^q)$ and

$$\hat{K}_{ij}(u,w) = qw^{q-1}(u^{q-1} + u^{q-2}w \cdots + w^{q-1})^{-\alpha} K_{ij}(u^q, w^q) \quad (i, j = 1, 2).$$

The existence and uniqueness results and the smoothness behavior of solutions $\hat{y}, \hat{z}$ of the system (2.5) may be obtained from the corresponding discussions of the classical theory of Volterra integral equations with weakly singular kernels from [5] (see e.g. Theorems 6.1.6 and 6.1.14 for further details).

3. The numerical treatments

3.1. Description of the numerical method

We now turn our attention to obtaining a Chebyshev spectral method for the system (2.5) on the standard interval $[-1, 1]$. Hence, we employ the transformation

$$x = \frac{2}{\sqrt[q]{T}} w - 1, \quad -1 \leq x \leq \upsilon, \qquad \upsilon = \frac{2}{\sqrt[q]{T}} u - 1, \quad -1 \leq \upsilon \leq 1, \tag{3.1}$$

to rewrite the system (2.5) as follows

$$\begin{cases} \tilde{y}(\upsilon) = \tilde{f}(\upsilon) + \int_{-1}^{\upsilon} (\upsilon - x)^{-\alpha} \tilde{K}_{11}(\upsilon, x) \tilde{y}(x) dx + \int_{-1}^{\upsilon} (\upsilon - x)^{-\alpha} \tilde{K}_{12}(\upsilon, x) \tilde{z}(x) dx, \\ 0 = \tilde{g}(\upsilon) + \int_{-1}^{\upsilon} (\upsilon - x)^{-\alpha} \tilde{K}_{21}(\upsilon, x) \tilde{y}(x) dx + \int_{-1}^{\upsilon} (\upsilon - x)^{-\alpha} \tilde{K}_{22}(\upsilon, x) \tilde{z}(x) dx, \end{cases} \tag{3.2}$$

where $\tilde{y}(\upsilon) = \hat{y}(\frac{\sqrt[q]{T}}{2}(\upsilon+1))$, $\tilde{z}(\upsilon) = \hat{z}(\frac{\sqrt[q]{T}}{2}(\upsilon+1))$, $\tilde{f}(\upsilon) = \hat{f}(\frac{\sqrt[q]{T}}{2}(\upsilon+1))$, $\tilde{g}(\upsilon) = \hat{g}(\frac{\sqrt[q]{T}}{2}(\upsilon+1))$ and

$$\tilde{K}_{ij}(\upsilon, x) = \left(\frac{\sqrt[q]{T}}{2}\right)^{1-\alpha} \hat{K}_{ij}\left(\frac{\sqrt[q]{T}}{2}(\upsilon+1), \frac{\sqrt[q]{T}}{2}(x+1)\right).$$

It is well-known that, in the Chebyshev collocation method we seek the solutions $\tilde{y}_N$ and $\tilde{z}_N$ of the form

$$\begin{aligned} \tilde{y}_N &= I_N(\tilde{y}(\upsilon)) = \sum_{k=0}^{N} \tilde{y}(\upsilon_k) L_k(\upsilon), \\ \tilde{z}_N &= I_N(\tilde{z}(\upsilon)) = \sum_{k=0}^{N} \tilde{z}(\upsilon_k) L_k(\upsilon), \end{aligned} \tag{3.3}$$

where $\upsilon_k = -\cos(\frac{(2k+1)\pi}{2N+2})$, $(k = 0, 1, \ldots, N)$ are the Gauss quadrature points and L_k are the interpolating Lagrange polynomials

$$L_k(\upsilon) = \frac{T_{N+1}(\upsilon)}{(\upsilon - \upsilon_k) T'_{N+1}(\upsilon)}, \quad k = 0, 1, \ldots, N, \tag{3.4}$$

such that T_{N+1} is the $(N+1)$th-order Chebyshev polynomial.

We now fix the value of x for general kernels $\tilde{K}_{ij}(\upsilon, x)$ and choose $x = \upsilon_k$, then the kernels $\tilde{K}_{ij}(\upsilon, x)$ can be approximated by univariate Lagrange interpolating polynomial as follows:

$$I_N(\tilde{K}_{ij}(\upsilon, x)) = \sum_{k=0}^{N} \tilde{K}_{ij}(\upsilon, \upsilon_k) L_k(x), \quad \forall i, j = 1, 2. \tag{3.5}$$

Substituting the relations (3.3) and (3.5) into (3.2) and inserting the collocation points υ_k in the obtained equation, lead to the following system of linear equations with $2N+2$ unknown values $\tilde{y}(\upsilon_0), \ldots, \tilde{y}(\upsilon_N)$ and $\tilde{z}(\upsilon_0), \ldots, \tilde{z}(\upsilon_N)$

$$\begin{cases} \tilde{y}(\upsilon_k) = \tilde{f}(\upsilon_k) + \int_{-1}^{\upsilon_k} (\upsilon_k - x)^{-\alpha} I_N(\tilde{K}_{11}(\upsilon_k, x)) \tilde{y}_N(x) + \int_{-1}^{\upsilon_k} (\upsilon_k - x)^{-\alpha} I_N(\tilde{K}_{12}(\upsilon_k, x)) \tilde{z}_N(x) dx, \\ 0 = \tilde{g}(\upsilon_k) + \int_{-1}^{\upsilon_k} (\upsilon_k - x)^{-\alpha} I_N(\tilde{K}_{21}(\upsilon_k, x)) \tilde{y}_N(x) dx + \int_{-1}^{\upsilon_k} (\upsilon_k - x)^{-\alpha} I_N(\tilde{K}_{22}(\upsilon_k, x)) \tilde{z}_N(x) dx. \end{cases} \tag{3.6}$$

Solving the linear system (3.6), the approximate solution of system (3.2) is determined at the collocation points as well as at the arbitrary points in the interval $[-1, 1]$ by (3.3).

3.2. Convergence phenomenon

Before giving our strategy for convergence phenomenon in the weighted L^2-norm, we first introduce some lemmas which are usually required to obtain the convergence results:

Lemma 1 (*From [20,21]*). *Let $\mathcal{I}_N$ be a linear operator from $C^{k,\beta}([-1, 1])$ to $\mathcal{P}_N$, then for any non-negative integer k and $\beta \in [0, 1]$, there exists a positive constant $C_{k,\beta} > 0$, such that*

$$\forall f \in C^{k,\beta}([-1, 1]), \ \exists \mathcal{I}_N f \in \mathcal{P}_N, \ s.t, \quad \|f - \mathcal{I}_N f\|_{L^\infty} \le C_{k,\beta} \, N^{-(k+\beta)} \|f\|_{C^{k,\beta}([-1,1])}.$$

Lemma 2 (*From [22]*). *Assume that $\{L_j(x)\}_{j=0}^{N}$ be Lagrange interpolation polynomials with the Chebyshev Gauss/Gauss–Radau/Gauss–Lobatto points $\{x_j\}_{j=0}^{N}$, then*

$$\|I_N\|_{L^\infty} = \max_{x \in (-1,1)} \sum_{i=0}^{N} |L_i(x)| = \mathcal{O}(\log N).$$

We denote the collocation error functions by

$$e_1(\upsilon) = (\tilde{y}_N(\upsilon) - \tilde{y}(\upsilon)), \qquad e_2(\upsilon) = (\tilde{z}_N(\upsilon) - (\tilde{z}(\upsilon))).$$

Rewriting the first equation of system (3.6) yields:

$$\tilde{y}(\upsilon_k) = \tilde{f}(\upsilon_k) + \sum_{n=0}^{N}\sum_{l=n}^{N}(a_{nl} + b_{nl})W_{nl}(\upsilon_k), \tag{3.7}$$

where

$$a_{nl} = \begin{cases} \tilde{y}_n(\tilde{K}_{11})_n, & n = l, \\ \tilde{y}_n(\tilde{K}_{11})_l + \tilde{y}_l(\tilde{K}_{11})_n, & n \neq l, \end{cases} \qquad b_{nl} = \begin{cases} \tilde{z}_n(\tilde{K}_{12})_n, & n = l, \\ \tilde{z}_n(\tilde{K}_{12})_l + \tilde{z}_l(\tilde{K}_{12})_n, & n \neq l, \end{cases}$$

and

$$(\tilde{K}_{ij})_n = \tilde{K}_{ij}(\upsilon_k, \upsilon_n), \quad (i, j = 1, 2), \qquad \tilde{y}_n = \tilde{y}(\upsilon_n),$$
$$W_{nl}(\upsilon_k) = \int_{-1}^{\upsilon_k}(\upsilon_k - x)^{-\alpha}L_n(x)L_l(x)dx.$$

With these notations, the Eq. (3.7) can be written as:

$$\begin{aligned} \tilde{y}(\upsilon_k) &= \tilde{f}(\upsilon_k) + \int_{-1}^{\upsilon_k}(\upsilon_k - x)^{-\alpha}\Big(\tilde{K}_{11}(\upsilon_k, x)e_1(x) + \tilde{K}_{12}(\upsilon_k, x)e_2(x)\Big)dx \\ &\quad + \int_{-1}^{\upsilon_k}(\upsilon_k - x)^{-\alpha}\Big(\tilde{K}_{11}(\upsilon_k, x)\tilde{y}(x) + \tilde{K}_{12}(\upsilon_k, x)\tilde{z}(x)\Big)dx + Z_1(\upsilon_k) + Z_2(\upsilon_k), \end{aligned} \tag{3.8}$$

such that

$$\begin{aligned} Z_1(\upsilon_k) &= \sum_{n=0}^{N}\sum_{l=n}^{N} a_{nl}W_{nl}(\upsilon_k) - \int_{-1}^{\upsilon_k}(\upsilon_k - x)^{-\alpha}\tilde{K}_{11}(\upsilon_k, x)I_N(\tilde{y}(x))dx, \\ Z_2(\upsilon_k) &= \sum_{n=0}^{N}\sum_{l=n}^{N} b_{nl}W_{nl}(\upsilon_k) - \int_{-1}^{\upsilon_k}(\upsilon_k - x)^{-\alpha}\tilde{K}_{12}(\upsilon_k, x)I_N(\tilde{z}(x))dx. \end{aligned} \tag{3.9}$$

If we multiply the Eq. (3.8) by $L_k(\upsilon)$ and sum up from 0 to N, we obtain

$$\begin{aligned} \tilde{y}_N(\upsilon) &= \tilde{f}_N(\upsilon) + \left\{\int_{-1}^{\upsilon}(\upsilon - x)^{-\alpha}\Big(\tilde{K}_{11}(\upsilon, x)e_1(x) + \tilde{K}_{12}(\upsilon, x)e_2(x)\Big)dx\right\}_N \\ &\quad + \left\{\int_{-1}^{\upsilon}(\upsilon - x)^{-\alpha}\Big(\tilde{K}_{11}(\upsilon, x)\tilde{y}(x) + \tilde{K}_{12}(\upsilon, x)\tilde{z}(x)\Big)dx\right\}_N + I_N(Z_1(\upsilon)) + I_N(Z_2(\upsilon)). \end{aligned} \tag{3.10}$$

Subtracting (3.10) from the first equation of (3.2), we get

$$\begin{aligned} e_1(\upsilon) &= \left\{\int_{-1}^{\upsilon}(\upsilon - x)^{-\alpha}\Big(\tilde{K}_{11}(\upsilon, x)e_1(x) + \tilde{K}_{12}(\upsilon, x)e_2(x)\Big)dx\right\}_N \\ &\quad + W_1 + W_2 + W_3 + I_N(Z_1(\upsilon)) + I_N(Z_2(\upsilon)), \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} W_1 &= \left\{\int_{-1}^{\upsilon}(\upsilon - x)^{-\alpha}\tilde{K}_{11}(\upsilon, x)\tilde{y}(x)dx\right\}_N - \int_{-1}^{\upsilon}(\upsilon - x)^{-\alpha}\tilde{K}_{11}(\upsilon, x)\tilde{y}(x)dx, \\ W_2 &= \left\{\int_{-1}^{\upsilon}(\upsilon - x)^{-\alpha}\tilde{K}_{12}(\upsilon, x)\tilde{z}(x)dx\right\}_N - \int_{-1}^{\upsilon}(\upsilon - x)^{-\alpha}\tilde{K}_{12}(\upsilon, x)\tilde{z}(x)dx, \\ W_3 &= \tilde{f}_N(\upsilon) - \tilde{f}(\upsilon). \end{aligned}$$

The Eq. (3.11) may be rewritten as follows:

$$\begin{aligned} e_1(\upsilon) &= \int_{-1}^{\upsilon}(\upsilon - x)^{-\alpha}\Big(\tilde{K}_{11}(\upsilon, x)e_1(x) + \tilde{K}_{12}(\upsilon, x)e_2(x)\Big)dx \\ &\quad + W_1 + W_2 + W_3 + W_4 + W_5 + I_N(Z_1(\upsilon)) + I_N(Z_2(\upsilon)), \end{aligned} \tag{3.12}$$

where

$$W_4=\left\{\int_{-1}^{\upsilon}(\upsilon-x)^{-\alpha}\tilde{K}_{11}(\upsilon,x)e_1(x)dx\right\}_N-\int_{-1}^{\upsilon}(\upsilon-x)^{-\alpha}\tilde{K}_{11}(\upsilon,x)e_1(x)dx,$$

$$W_5=\left\{\int_{-1}^{\upsilon}(\upsilon-x)^{-\alpha}\tilde{K}_{12}(\upsilon,x)e_2(x)dx\right\}_N-\int_{-1}^{\upsilon}(\upsilon-x)^{-\alpha}\tilde{K}_{12}(\upsilon,x)e_2(x)dx.$$

Now, from the second equation of (3.2), we get

$$\left\{\int_{-1}^{\upsilon}(\upsilon-x)^{-\alpha}\Big(\tilde{K}_{21}(\upsilon,x)\tilde{y}(x)+\tilde{K}_{22}(\upsilon,x)\tilde{z}(x)\Big)dx\right\}_N=-\tilde{g}_N(\upsilon). \tag{3.13}$$

For the second equation of (3.6), we proceed a similar procedure as outlined for obtaining the relation (3.10), and then insert (3.13) into the resulting equation, so we obtain

$$0=\int_{-1}^{\upsilon}(\upsilon-x)^{-\alpha}\Big(\tilde{K}_{21}(\upsilon,x)e_1(x)+\tilde{K}_{22}(\upsilon,x)e_2(x)\Big)dx+W_6+W_7+I_N(Z_3(\upsilon))+I_N(Z_4(\upsilon)), \tag{3.14}$$

where

$$W_6=\left\{\int_{-1}^{\upsilon}(\upsilon-x)^{-\alpha}\tilde{K}_{21}(\upsilon,x)e_1(x)dx\right\}_N-\int_{-1}^{\upsilon}(\upsilon-x)^{-\alpha}\tilde{K}_{21}(\upsilon,x)e_1(x)dx,$$

$$W_7=\left\{\int_{-1}^{\upsilon}(\upsilon-x)^{-\alpha}\tilde{K}_{22}(\upsilon,x)e_2(x)dx\right\}_N-\int_{-1}^{\upsilon}(\upsilon-x)^{-\alpha}\tilde{K}_{22}(\upsilon,x)e_2(x)dx,$$

and

$$Z_3(\upsilon_k)=\sum_{n=0}^{N}\sum_{l=n}^{N}c_{nl}W_{nl}(\upsilon_k)-\int_{-1}^{\upsilon_k}(\upsilon_k-x)^{-\alpha}\tilde{K}_{21}(\upsilon_k,x)I_N(\tilde{y}(x))dx,$$

$$Z_4(\upsilon_k)=\sum_{n=0}^{N}\sum_{l=n}^{N}d_{nl}W_{nl}(\upsilon_k)-\int_{-1}^{\upsilon_k}(\upsilon_k-x)^{-\alpha}\tilde{K}_{22}(\upsilon_k,x)I_N(\tilde{z}(x))dx,$$

such that

$$c_{nl}=\begin{cases}\tilde{y}_n(\tilde{K}_{21})_n, & n=l,\\ \tilde{y}_n(\tilde{K}_{21})_l+\tilde{y}_l(\tilde{K}_{21})_n, & n\neq l,\end{cases}\qquad d_{nl}=\begin{cases}\tilde{z}_n(\tilde{K}_{22})_n, & n=l,\\ \tilde{z}_n(\tilde{K}_{22})_l+\tilde{z}_l(\tilde{K}_{22})_n, & n\neq l.\end{cases}$$

Using a similar manner which is applied in Section 2 for obtaining (2.1), the Eq. (3.14) can be written as:

$$0=\int_{-1}^{\upsilon}\tilde{H}_{21}(\upsilon,x)e_1(x)dx+\int_{-1}^{\upsilon}\tilde{H}_{22}(\upsilon,x)e_2(x)dx+G_\alpha(F), \tag{3.15}$$

where $F=W_6+W_7+I_N(Z_3(\upsilon))+I_N(Z_4(\upsilon))$, and

$$G_\alpha(F)=\int_{-1}^{\upsilon}(\upsilon-x)^{\alpha-1}F(x)dx.$$

Differentiation (3.15) with respect to υ, yields a second kind integral equation as follows:

$$\tilde{H}_{21}(\upsilon,\upsilon)e_1(\upsilon)-\tilde{H}_{22}(\upsilon,\upsilon)e_2(\upsilon)=\int_{-1}^{\upsilon}\left(\frac{\partial\tilde{H}_{21}(\upsilon,x)}{\partial\upsilon}e_1(x)+\frac{\partial\tilde{H}_{22}(\upsilon,x)}{\partial\upsilon}e_2(x)\right)dx+G'_\alpha(F), \tag{3.16}$$

where $G'_\alpha(F)$ can be obtained using integration by parts to $G_\alpha(F)$ and $F(-1)=0$

$$G'_\alpha(F)=\int_{-1}^{\upsilon}(\upsilon-x)^{\alpha-1}F'(x)dx. \tag{3.17}$$

In this position, to obtain a matrix representation of the resulting equations, we rewrite the Eq. (3.16) as

$$\begin{aligned}-\tilde{H}_{21}(\upsilon,\upsilon)e_1(\upsilon)-\tilde{H}_{22}(\upsilon,\upsilon)e_2(\upsilon)&=\int_{-1}^{\upsilon}(\upsilon-x)^{-\alpha}\left((\upsilon-x)^{\alpha}\frac{\partial\tilde{H}_{21}(\upsilon,x)}{\partial\upsilon}e_1(x)\right)dx\\&\quad+\int_{-1}^{\upsilon}(\upsilon-x)^{-\alpha}\left((\upsilon-x)^{\alpha}\frac{\partial\tilde{H}_{22}(\upsilon,x)}{\partial\upsilon}e_2(x)\right)dx+G'_\alpha(F),\end{aligned} \tag{3.18}$$

and denote

$$\mathbf{H}(\upsilon,\upsilon)=\begin{pmatrix}1 & 0\\ -\tilde{H}_{21}(\upsilon,\upsilon) & -\tilde{H}_{22}(\upsilon,\upsilon)\end{pmatrix},$$

$$\mathbf{R}(\upsilon,\mathbf{x})=\begin{pmatrix}\tilde{K}_{11}(\upsilon,x) & \tilde{K}_{12}(\upsilon,x)\\ (\upsilon-x)^{\alpha}\dfrac{\partial\tilde{H}_{21}(\upsilon,x)}{\partial\upsilon} & (\upsilon-x)^{\alpha}\dfrac{\partial\tilde{H}_{22}(\upsilon,x)}{\partial\upsilon}\end{pmatrix},$$

$$\mathbf{E}(\upsilon)=\begin{pmatrix}e_1(\upsilon)\\ e_2(\upsilon)\end{pmatrix},\qquad \mathbf{M}=\begin{pmatrix}W_1+W_2+W_3+W_4+W_5+I_N(Z_1(\upsilon))+I_N(Z_2(\upsilon))\\ G'_{\alpha}(F)\end{pmatrix}.$$

With these notations, the corresponding matrix representation of the Eqs. (3.12) and (3.18) is given by

$$\mathbf{H}(\upsilon,\upsilon)\mathbf{E}(\upsilon)=\int_{-1}^{\upsilon}(\upsilon-x)^{-\alpha}\mathbf{R}(\upsilon,x)\mathbf{E}(\mathbf{x})dx+\mathbf{M}. \tag{3.19}$$

Following [10, Theorem (8.1.8)], we have $|\tilde{K}_{22}(\upsilon,\upsilon)|\ \geq\ k_0\ >\ 0$ on $[-1,1]$, moreover, from (2.2), $\tilde{H}_{21}(\upsilon,\upsilon)=\frac{\sin(\alpha\pi)}{\pi}\tilde{K}_{21}(\upsilon,\upsilon)$ and $\tilde{H}_{22}(\upsilon,\upsilon)=\frac{\sin(\alpha\pi)}{\pi}\tilde{K}_{22}(\upsilon,\upsilon)$, so the matrix $\mathbf{H}(\upsilon,\upsilon)$ is invertible and its inverse can be set in the form

$$\mathbf{H}^{-1}(\upsilon,\upsilon)=\begin{pmatrix}1 & 0\\ \dfrac{-\tilde{K}_{21}(\upsilon,\upsilon)}{\tilde{K}_{22}(\upsilon,\upsilon)} & \dfrac{-\pi}{\sin(\alpha\pi)\tilde{K}_{22}(\upsilon,\upsilon)}\end{pmatrix}.$$

Multiplying the Eq. (3.19) by $\mathbf{H}^{-1}(\upsilon,\upsilon)$, gives

$$|\mathbf{E}(\upsilon)|\leq\Phi\int_{-1}^{\upsilon}(\upsilon-x)^{-\alpha}|\mathbf{E}(\mathbf{x})|dx+|\widehat{\mathbf{N}}|, \tag{3.20}$$

where $\Phi=\max_{-1\leq x<\upsilon\leq 1}|\mathbf{H}^{-1}(\upsilon,\upsilon)\mathbf{R}(\upsilon,\mathbf{x})|$ and $\widehat{\mathbf{N}}=\mathbf{H}^{-1}(\upsilon,\upsilon)\mathbf{M}$.

Employing the generalized Gronwall inequality [23], we can write

$$|\mathbf{E}(\upsilon)|\leq\Phi\int_{-1}^{\upsilon}(\upsilon-x)^{-\alpha}|\widehat{\mathbf{N}}(\mathbf{x})|dx+|\widehat{\mathbf{N}}|. \tag{3.21}$$

It can be seen from the generalized Hardy's inequality (see e.g. Lemma (3.8) from [23]) that

$$\|\mathbf{E}\|_{\mathbf{L}^2_{\omega}(-\mathbf{1},\mathbf{1})}\leq C\|\widehat{\mathbf{N}}\|_{\mathbf{L}^2_{\omega}(-\mathbf{1},\mathbf{1})}. \tag{3.22}$$

It remains to derive a bound for the global error and obtain the order of convergence of the proposed method. In the continuation of the paper, for simplifying the expressions, we denote $\|\cdot\|_{L^2_{\omega}(-1,1)}$ by $\|\cdot\|$ and try to obtain the error bounds step by step:

Step 1: In this position we use some previously given lemmas and also some known lemmas and inequalities from [24, 23] to achieve the error bounds for $W_i,\ (i=1,\ldots,5)$.

Since I_N indicates the interpolation operator, then we have

$$I_N\phi(x)=\phi(x),\qquad\forall\phi(x)\in\mathcal{P}_N.$$

Now, we observe that

$$\begin{aligned}\|W_4\|&=\|I_N\Gamma e_1-\Gamma e_1\|=\|I_N\Gamma e_1-I_N(\mathcal{I}_{\mathcal{N}}\Gamma e_1)+\mathcal{I}_{\mathcal{N}}\Gamma e_1-\Gamma e_1\|\\&\leq\|I_N\|\,\|\Gamma e_1-\mathcal{I}_{\mathcal{N}}\Gamma e_1\|+\|\mathcal{I}_{\mathcal{N}}\Gamma e_1-\Gamma e_1\|\\&\leq(\|I_N\|_{L^\infty}+1)\|\Gamma e_1-\mathcal{I}_{\mathcal{N}}\Gamma e_1\|_{L^\infty},\end{aligned} \tag{3.23}$$

where $\mathcal{I}_{\mathcal{N}}$ is defined in Lemma 1, and

$$\Gamma e_1=\int_{-1}^{\upsilon}(\upsilon-x)^{-\alpha}\tilde{K}_{11}(\upsilon,x)e_1(x)dx.$$

In order to obtain a bound for (3.23), we first use Lemma 1 for $k=0$ and Lemma 2. Then employing the Lemma (3.5) from [23] and the inequality (5.5.28) from [24], give us:

$$\begin{aligned}\|W_4\|&\leq C(\log N+1)N^{-\beta}\|\Gamma e_1\|_{C^{0,\beta}([-1,1])}\leq C\log NN^{-\beta}\|e_1\|_{L^\infty}\\&\leq C\log NN^{\frac{1}{2}-\beta-m}|\tilde{y}|_{H^{m,N}_{\omega}(-1,1)},\end{aligned} \tag{3.24}$$

where

$$|\tilde{y}|_{H_\omega^{m,N}(-1,1)} = \left(\sum_{j=\min(m,N+1)}^{m} \|\tilde{y}^{(j)}\|^2_{L^2_\omega(-1,1)} \right)^{\frac{1}{2}}. \tag{3.25}$$

As a similar manner

$$\|W_5\| \le C \log N N^{\frac{1}{2}-\beta-m} |\tilde{z}|_{H_\omega^{m,N}(-1,1)}.$$

It follows from (3.24) that

$$\|W_1\| \le C(\log N + 1) N^{-\beta} \|\Gamma \tilde{y}\|_{C^{0,\beta}([-1,1])} \le C \log N N^{-\beta} \|\tilde{y}\|_{L^\infty},$$

and

$$\|W_2\| \le C \log N N^{-\beta} \|\tilde{z}\|_{L^\infty}.$$

Also, using the inequality (5.5.22) from [24], we have

$$\|W_3\| \le C N^{-m} |\tilde{f}|_{H_\omega^{m,N}(-1,1)}.$$

Step 2: In this step, our aim is to estimate $I_N(Z_1(\upsilon))$ and $I_N(Z_2(\upsilon))$ by using Lemma 2, appropriately

$$\|I_N(Z_1(\upsilon))\| \le \max_{0\le k\le N} |Z_1(\upsilon_k)| \, \|I_N\|_{L^\infty} \le \max_{0\le k\le N} |Z_1(\upsilon_k)| \log N. \tag{3.26}$$

Furthermore, from (3.9) we have

$$\begin{aligned} |Z_1(\upsilon_k)| &= \left| \int_{-1}^{\upsilon_k} (\upsilon_k - x)^{-\alpha} \left(I_N\left(\tilde{K}_{11}(\upsilon_k, x)\right) - \tilde{K}_{11}(\upsilon_k, x) \right) \tilde{y}_N(x) dx \right| \\ &\le C \|I_N\left(\tilde{K}_{11}(\upsilon_k, x)\right) - \tilde{K}_{11}(\upsilon_k, x)\|_{L^\infty} \, \|\tilde{y}_N\|_{L^\infty} \int_{-1}^{\upsilon_k} (\upsilon_k - x)^{-\alpha} dx, \end{aligned}$$

such that using the transformation (3.1), we can write

$$\int_{-1}^{\upsilon_k} (\upsilon_k - x)^{-\alpha} dx = (\upsilon_k + 1)^{(1-\alpha)} B(1-\alpha, 1),$$

where $B(\cdot,\cdot)$ denotes the *Euler Beta function* (see [5, p. 354]).
It follows from the inequality (5.5.28) from [24]

$$|Z_1(\tau_k)| \le (\upsilon_k + 1)^{(1-\alpha)} B(1-\alpha, 1) N^{\frac{1}{2}-m} |\tilde{K}_{11}(\upsilon_k, x)|_{H_\omega^{m,N}(-1,1)} \|\tilde{y}_N\|_{L^\infty}. \tag{3.27}$$

Using (3.27), the following relation for (3.26) holds

$$\|I_N(Z_1(\upsilon))\| \le C \log N N^{\frac{1}{2}-m} \Phi_{11} \|\tilde{y}_N\|_{L^\infty}, \tag{3.28}$$

where

$$\Phi_{ij} = B(1-\alpha, 1) \max_{0\le k\le N} (\upsilon_k + 1)^{(1-\alpha)} |\tilde{K}_{ij}(\upsilon_k, x)|_{H_\omega^{m,N}(-1,1)}, \quad (i, j = 1, 2).$$

Consequently, we observe that

$$\|I_N(Z_2(\upsilon))\| \le C \log N N^{\frac{1}{2}-m} \Phi_{12} \|\tilde{z}_N\|_{L^\infty}.$$

Step 3: Here, we should find a bound for $|G'_\alpha(F)|$ using the suitable inequalities as well as the previously obtained bounds. For this purpose, we estimate the Eq. (3.17) as:

$$|G'_\alpha(F)| \le \int_{-1}^{\upsilon} (\upsilon - x)^{\alpha-1} |F'(s)| ds.$$

Considering the generalized Hardy's inequality, it can also be shown that

$$\|G'_\alpha(F)\| \le C\|F'(s)\| \le C\Big(\|I'_N(Z_3(\upsilon))\| + \|I'_N(Z_4(\upsilon))\| + \|W'_6\| + \|W'_7\| \Big). \tag{3.29}$$

Indeed, applying the inequality (5.5.25) from [24], we have

$$\|W'_6\| \le C N^{1-m} |\tilde{\Gamma} e_1|_{H_\omega^{m,N}(-1,1)},$$

where $\tilde{\Gamma} e_1 = \int_{-1}^{\upsilon} (\upsilon - x)^{-\alpha} \tilde{K}_{21}(\upsilon, x) e_1(x) dx$.

Setting $m = 1$ in (3.25), we can write

$$\|W_6'\| \le C \left\| \frac{\partial \tilde{\Gamma} e_1}{\partial \upsilon} \right\|,$$

such that by using integration by parts and the generalized Hardy's inequality

$$\left\| \frac{\partial \tilde{\Gamma} e_1}{\partial \upsilon} \right\| \le C(\|e_1\| + \|e_1'\|).$$

Now, utilizing the inequalities (5.5.22) and (5.5.25) from [24], we get the following result

$$\|W_6'\| \le C(N^{-m} + N^{1-m})|\tilde{y}|_{H_\omega^{m,N}(-1,1)}.$$

As a similar manner,

$$\|W_7'\| \le C(\|e_2\| + \|e_2'\|) \le C(N^{-m} + N^{1-m})|\tilde{z}|_{H_\omega^{m,N}(-1,1)}.$$

On the other hand, using the inequality (5.5.4) from [24] and the relation (3.28), we have

$$\|I_N'(Z_3(\upsilon))\| \le CN^2\|I_N(Z_3(\upsilon))\| \le \log N N^{\frac{5}{2}-m}\Phi_{21}\|\tilde{y}_N\|_{L^\infty},$$

and

$$\|I_N'(Z_4(\upsilon))\| \le CN^2\|I_N(Z_4(\upsilon))\| \le \log N N^{\frac{5}{2}-m}\Phi_{22}\|\tilde{z}_N\|_{L^\infty}.$$

Finally, the above estimates together with (3.22), lead to the following main theorem which reveals the convergence results of the presented scheme:

Theorem 1. *Consider the index-1 weakly singular integral algebraic equation (2.3) and its transformed representation (3.2). Let $\tilde{D} = \{(t, s) : -1 \le x \le \upsilon \le T\}$, and suppose the following holds*

(a) *The given functions $\tilde{K}_{1j}$ $(i, j = 1, 2)$ and $\tilde{f}$ in (3.2) belong to $C^m(\tilde{D})$ and $C^m[-1, 1]$, respectively.*
(b) *$\tilde{K}_{2j} \in C^{m+1}(\tilde{D})$ $(i, j = 1, 2)$ and $\tilde{g} \in C^{m+1}[-1, 1]$ with $\tilde{g}(-1) = 0$.*
(c) *$\tilde{K}_{22}$ satisfies the condition $|\tilde{K}_{22}(\upsilon, \upsilon)| > 0$.*
(d) *$(\tilde{y}_N, \tilde{z}_N)$ denotes the spectral collocation approximation for the exact solution $(\tilde{y}, \tilde{z})$ of (3.2) based on the Chebyshev Gauss collocation points which is given by (3.3) and (3.6).*

Then for sufficiently large N,

$$\|\tilde{y} - \tilde{y}_N\|_{L^2_w(-1,1)} \simeq \mathcal{O}\left(N^{\frac{1}{2}-m}\log N\right),$$

$$\|\tilde{z} - \tilde{z}_N\|_{L^2_w(-1,1)} \simeq \mathcal{O}\left(N^{\frac{5}{2}-m}\log N\right).$$

4. Numerical experiments

In this section, we consider some numerical examples in order to illustrate the validity of the proposed technique. Coordinate transformations (2.4) and variable transformations (3.1) are used to change the WSIAEs system into a new system such that its solutions have better regularity. All the computations were performed using software Mathematica®. For analyzing the behavior of the error representations, we define the weighted L^2-norm by

$$\|e\|_{L^2_w(-1,1)} = \left(\int_{-1}^{1} |e|^2 w(t)dt\right)^{\frac{1}{2}},$$

where $w(t) = \frac{1}{\sqrt{1-t^2}}$ is the Chebyshev weight function.

Example 1. Consider the following index-1 WSIAEs system

$$AX(t) = G(t) + \int_0^t (t-s)^{-\frac{1}{2}}K(t, s)X(s)ds, \quad t \in [0, 1],$$

where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad K(t, s) = \begin{bmatrix} t^2 + 2 & t + s^3 \\ s^2 + 4 & t^2 + s^4 + 1 \end{bmatrix},$$

$$X(t) = \big(y(t), z(t)\big)^T, \qquad G(t) = \big(f(t), g(t)\big)^T,$$

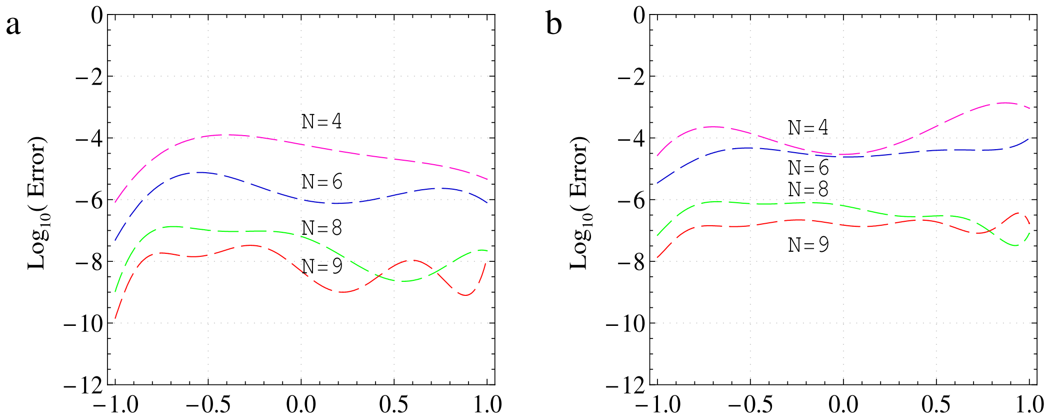

Fig. 1. (a) Plot of error functions of $\tilde{y}$ for different values of N in Example 1. (b) Plot of error functions of $\tilde{z}$ for different values of N in Example 1.

Table 1
L^2_w errors for Example 1.

N	4	5	6	7	8	9
$\Vert\tilde{y}-\tilde{y}_N\Vert_{L^2_w}$	1.64×10^{-4}	3.63×10^{-5}	9.43×10^{-6}	4.92×10^{-7}	2.74×10^{-7}	5.54×10^{-8}
$\Vert\tilde{z}-\tilde{z}_N\Vert_{L^2_w}$	1.71×10^{-3}	2.60×10^{-4}	8.08×10^{-5}	4.42×10^{-6}	9.01×10^{-7}	3.08×10^{-7}

and f, g are chosen such that the exact solution is

$$y(t)=\arctan\sqrt{t},\qquad z(t)=\frac{\exp t-1}{\sqrt{t}}.$$

Due to the first derivatives of the exact solution

$$y'(t)=\frac{t^{-\frac{1}{2}}}{2(1+t)},\qquad z'(t)=t^{-\frac{1}{2}}\left(\frac{\exp t-1}{2t}+\exp t\right),$$

we observe that $y'\sim t^{-\frac{1}{2}}$ and $z'\sim t^{-\frac{1}{2}}$ at $t\to0$. Since $\alpha=\frac{p}{q}=\frac{1}{2}$, then employing the transformation $t=u^2$ gives the smooth solution $\hat{y}=y(u^2)$ and $\hat{z}=z(u^2)$. Inserting $\hat{y},\hat{z}$ in the WSIAE and using transformation (3.1) yield

$$A\widetilde{X}(\upsilon)=\widetilde{G}(\upsilon)+\int_{-1}^{\upsilon}(\upsilon-x)^{-\frac{1}{2}}\widetilde{K}(\upsilon,x)\widetilde{X}(x)dx,\quad \upsilon\in[-1,1], \tag{4.1}$$

where $\widetilde{X}(\upsilon)=(\tilde{y},\tilde{z})^T$, $\widetilde{G}(\upsilon)=(\tilde{f},\tilde{g})^T$ and

$$\widetilde{K}(\upsilon,x)=\begin{bmatrix}\widetilde{K}_{11}(\upsilon,x) & \widetilde{K}_{12}(\upsilon,x)\\ \widetilde{K}_{21}(\upsilon,x) & \widetilde{K}_{22}(\upsilon,x)\end{bmatrix}.$$

Let $\widetilde{X}_N=(\tilde{y}_N,\tilde{z}_N)$ denote the approximation of the exact solution $\widetilde{X}=(\tilde{y},\tilde{z})$ that is given by (3.3). We apply the proposed Chebyshev collocation scheme for the system (4.1) and report the weighted L^2-norm of errors for several values of N in Table 1. The error behaviors on $[-1,1]$ for different values of N are also represented in Fig. 1.

Example 2.

$$AX(t)=G(t)+\int_0^t(t-s)^{-\frac{1}{3}}K(t,s)X(s)ds,\quad t\in[0,1],$$

$$K(t,s)=\begin{bmatrix}t+s+2 & ts\\ (t+s)^2 & 1+st^2\end{bmatrix},$$

$$X(t)=\big(y(t),z(t)\big)^T,\qquad G(t)=\big(f(t),g(t)\big)^T.$$

Let f, g be chosen such that the exact solution is $y(t)=\sinh t^{\frac{2}{3}}$, $z(t)=t^{\frac{1}{3}}$. It is easy to check that $y'\sim t^{-\frac{1}{3}}$ and $z'\sim t^{-\frac{2}{3}}$ at $t\to0$. So, according to the transformation (2.4) and $\alpha=\frac{p}{q}=\frac{1}{3}$, the smooth solutions $\hat{y}=\sinh t^2$ and $\hat{z}=t$ are obtained.

The weighted L^2-norm of errors and the error behaviors on $[-1,1]$ for different values of N are reported in Table 2 and Fig. 2.

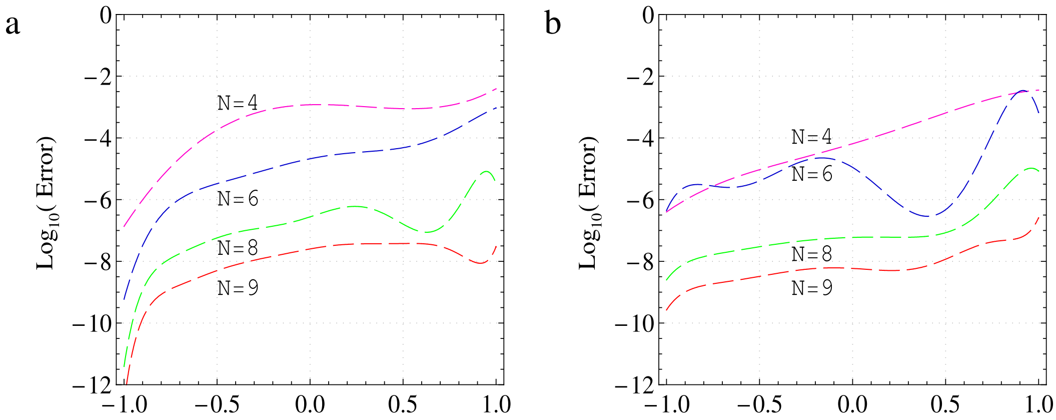

Fig. 2. (a) Plot of error functions of $\tilde{y}$ for different values of N in Example 2. (b) Plot of error functions of $\tilde{z}$ for different values of N in Example 2.

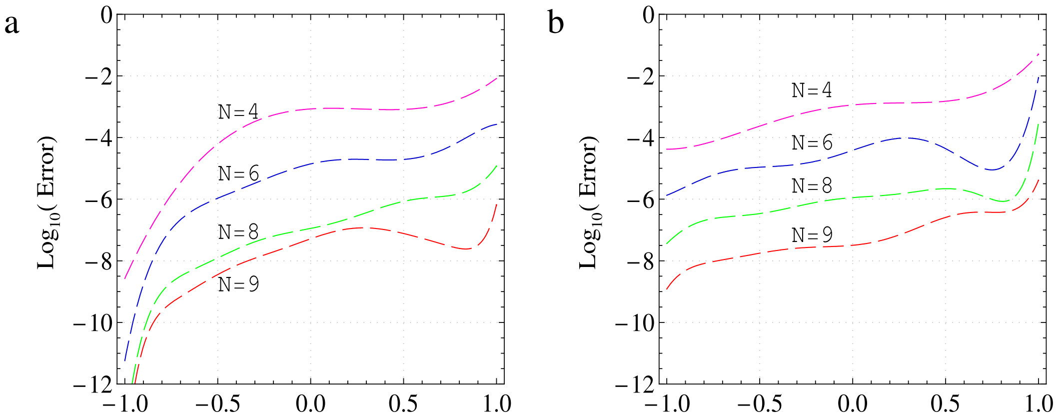

Fig. 3. (a) Plot of error functions of $\tilde{y}$ for different values of N in Example 3. (b) Plot of error functions of $\tilde{z}$ for different values of N in Example 3.

Table 2
L^2_w errors for Example 2.

N	4	5	6	7	8	9
$\Vert\tilde{y}-\tilde{y}_N\Vert_{L^2_w}$	2.90×10^{-3}	1.91×10^{-3}	5.57×10^{-4}	4.44×10^{-5}	3.54×10^{-6}	6.85×10^{-8}
$\Vert\tilde{z}-\tilde{z}_N\Vert_{L^2_w}$	2.73×10^{-3}	1.24×10^{-3}	1.10×10^{-3}	8.30×10^{-5}	6.38×10^{-6}	1.09×10^{-7}

Table 3
L^2_w errors for Example 3.

N	4	5	6	7	8	9
$\Vert\tilde{y}-\tilde{y}_N\Vert_{L^2_w}$	5.00×10^{-3}	1.92×10^{-3}	1.84×10^{-4}	9.19×10^{-5}	5.61×10^{-6}	2.41×10^{-7}
$\Vert\tilde{z}-\tilde{z}_N\Vert_{L^2_w}$	1.94×10^{-2}	1.05×10^{-2}	1.13×10^{-3}	7.78×10^{-4}	5.19×10^{-5}	1.61×10^{-6}

Example 3. As a third test problem, consider the following system of index-1 WSIAEs

$$AX(t) = G(t) + \int_0^t (t-s)^{-\frac{1}{4}} K(t,s)X(s)ds, \quad t \in [0,1],$$

$$K(t,s) = \begin{bmatrix} e^{\left(s^{\frac{1}{2}}+t\right)}(t^2+s^4+3) & \cos s^{\frac{1}{4}}(t+s) \\ e^{\left(s^{\frac{1}{2}}+t^2+1\right)}(t+s) & \sin\left(s^{\frac{1}{4}}+1\right)(st+1) \end{bmatrix},$$

$$X(t) = \big(y(t), z(t)\big)^T, \qquad G(t) = \big(f(t), g(t)\big)^T,$$

and f, g are chosen such that the exact solution is $y(t) = \exp(\sqrt{t}),\ z(t) = \sin(\sqrt[4]{t})$.

Table 3 and Fig. 3 represent the error behaviors of the computed solutions using the proposed collocation method.

5. Conclusion

This paper studied the theoretical and numerical treatments of weakly singular Volterra IAEs systems. We analyzed a spectral Chebyshev collocation approximation for the WSIAEs with index-1. The convergence analysis was included and it was shown that the numerical errors decay exponentially in the weighted L^2-norm.

Here, we considered the case when the underling solutions of the WSIAEs were not sufficiently smooth. To overcome this difficulty, we used some coordinate transformation to change the equations into a new WSIAEs. In our future work, we will investigate the approximate solution of WSIAEs with non-smooth given functions.

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