Computers and Mathematics with Applications 64 (2012) 2478–2483

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

λ-matrix formulation applied to the Hertz contact problem with finite friction

N. Shayanfar, M. Hadizadeh *

Department of Mathematics, K. N. Toosi University of Technology, P.O. Box 16315-1618, Tehran, Iran

ARTICLE INFO

Article history:
Received 7 February 2012
Received in revised form 9 April 2012
Accepted 31 May 2012

Keywords:
Hertz contact problem
Abel's integral equation
λ-matrix
Matrix decomposition
Numerical treatments

ABSTRACT

In this paper, we are concerned with coupled Volterra integral equations for rigid bodies in the frame of the Hertz contact model. Considering an appropriate integral operator as the variable of the matrix polynomial, a λ-matrix corresponding to the Hertz model is determined. Knowledge of the Smith decomposition forms of the λ-matrices enables us to characterize a single integral equation which recovers the normal stress as well as the compliance factor. The presented model shows a reliable behavior and an acceptable accuracy of the λ-matrix approach in comparison to the recent results in the literature.

1. Introduction

Contact mechanics is the study of the deformation of solids that touch each other at one or more points. Classical contact mechanics is most notably associated with Heinrich Hertz. The physical and mathematical formulation of the subject is built upon the mechanics of materials and continuum mechanics and focuses on computations involving elastic, viscoelastic and plastic bodies in static or dynamic contact.

It is well known that the contact force between two elastic bodies can be computed through the Hertzian classical model [1]. The elastic contact problem with the Coulomb friction law in the region of contact can be formulated in various models which generally require different numerical methods for their solution. In 1968, Spence [2] obtained the mathematical solution of the elastic contact problem in the case of fully adhesive contact for indentors of polynomial shape. Finite friction between the surfaces in contact is more realistic than complete adhesion. With this practical assumption, in 1971, Noble and Spence proposed an integral equation formulation for the Hertz contact problem [3]. A minimization problem with a complementarity constraint has also been considered for a variational inequality formulation in terms of stress and displacement [4]. In 1982, Bryant and Keer [5] computed the complete solution in the whole half-space for the field of stresses and displacements. Alternative solutions for contact problem were also given by Sackfield and Hills in [6,7], and Kikuchi and Oden in [8], using a finite element method. Moreover, in [9,10] some algorithms have been developed for solving the dynamic contact problems.

This paper is concerned with the integral equation formulation of the Hertz contact problem, which was first proposed in [3]. Spence suggested an iterative numerical method with the unknown free point of separating slip from no slip [11], and recently, Gauthier et al. [12] analyzed the problem with the mixed linear complementarity problem directly by solving in terms of the stress and displacement for the unknown free point. The aim of this paper is to present an efficient procedure for solving the problem, which is mainly connected with the λ-matrices. The connection of the λ-matrix equation and the

* Corresponding author. Fax: +98 2122853650.
E-mail address: hadizadeh@kntu.ac.ir (M. Hadizadeh).

0898-1221/$ – see front matter
doi:10.1016/j.camwa.2012.05.022

system of integral equations is determined and a constructive method for solving the contact problem will be given based on the Smith form of the corresponding λ-matrix.

λ-matrix is one of the most interesting topics in the numerical linear algebra which was studied by several authors [13,14], especially with regard to its applications such as investigation of singular equations [15], eigenvalues of matrix polynomials [16], signal processing [17] and analysis of linear feedback systems [18]. The factorization of λ-matrices also has various applications, the most common one involves solving the system of differential equations [13,19]. Another important application of the Smith form concerns the study of the algebraic structural properties of systems in linear control theory [19–21].

The organization of the paper is as follows. Section 2 is devoted to the formulation of the Hertz contact problem as a system of coupled integral equations, which is the same model as described in [12]. In this model, the normal compliance of the surface is the main topic of our analysis. Then, supposing the integral operators as the variables of a multivariable λ-matrix, a λ-matrix equation is derived, and a procedure based on the decomposition form of λ-matrices is constructed. The main idea, which is designed to cope with the multivariable λ-matrices, is proposed in Section 3. Using this formulation, an independent integral equation for investigating the normal stress as well as the normal compliance factor is obtained. Finally, in Section 4, some numerical results are reported which confirm the theoretical results of the paper, as well as the existing results reported in [11,12].

2. The Hertz contact problem

In this section, the mathematical model of the Hertz contact problem is developed and an equivalent λ-matrix equation is obtained. (Further details regarding the analysis of the considered model can be found in [12]). We will show that using the λ-matrix decomposition, an independent integral equation for the model is derived.

2.1. Formulation as a system of integral equations

In continuum mechanics, stress is a measure of the internal forces acting within a deformable body. Quantitatively, it is a measure of the average force per unit area of a surface within the body on which internal forces act. These internal forces are produced between the particles in the body as a reaction to external forces applied on the body. A shear stress is defined as a stress which is applied parallel or tangential to a face of a material, as opposed to a normal stress which is applied perpendicularly. In the case of Hertzian indentation, a system of coupled Volterra integral equations is formulated for normal and shear stress. To handle the surfaces at the contact, we apply an approach defining the surfaces using equations with constraints. When two solid surfaces are brought into close proximity of each other, they experience attractive forces between molecules. Physically, a realistic assumption is to suppose a finite coefficient of friction μ between the surfaces in contact.

This paper discusses the integral equation formulation of the punch indentation problem (with partial adhesion). The Coulomb friction law with coefficient μ in the region of contact is assumed to be satisfied with no slip occurring over the central circle, in which the radius c separates the zones of adhesion and slip. There is just one solution which provides a unique relation between the three parameters c, μ and $\gamma = (1 - 2\nu)/(2 - 2\nu)$, where ν is the Poisson ratio (for more details see [12]).

Following [22], we recall that when calculating local contact pressures, an elastic body can be regarded as an elastic half-space, so we consider an elastic half-space $z > 0$. Suppose that normal force p is applied monotonically to a rigid axisymmetric punch resting on an elastic half-space, starting from a state of zero stress. Gauthier et al. [12] derived the following coupled Abel's equations as a characterization of the Hertz contact problem with finite friction in terms of p and q, normal and shear stress, respectively:

$$\begin{cases} \displaystyle\int_x^1 \frac{tp(t)}{\sqrt{t^2-x^2}}dt - \gamma\left\{\int_0^1 q(t)dt - x\int_0^x \frac{q(t)}{\sqrt{x^2-t^2}}dt\right\} = 1, \\ \displaystyle\gamma\int_0^x \frac{tp(t)}{\sqrt{x^2-t^2}}dt - x\int_x^1 \frac{q(t)}{\sqrt{t^2-x^2}}dt + \frac{1}{x}\frac{d}{dx}\left[\int_0^x \frac{t^2u(t)}{\sqrt{x^2-t^2}}dt\right] = 0. \end{cases} \tag{2.1}$$

According to [11], the problem reduces to the solution for p and q on the interval $(0, 1)$. We look for a solution in which no slip takes place over a central circle $(0, c)$, where $(c < 1)$. Determining a relation between μ and γ for a fixed c is straightforward, and the existence of a unique point c has been proved in [11]. The complementarity conditions are derived as follows.

$$\begin{aligned} &u(x) = 0 \quad \text{and} \quad q(x) - \mu p(x) < 0, \quad x \in [0, c), \\ &u(x) < 0 \quad \text{and} \quad q(x) - \mu p(x) = 0, \quad x \in [c, 1], \end{aligned} \tag{2.2}$$

while in the outer annulus $(c, 1)$ slip takes place inward.

When no slip takes place, according to the conditions (2.2), the system of integral equations (2.1) can be written in the following coupled system of Abel integral equations of the first kind:

$$\begin{cases} \displaystyle\int_x^1 \frac{tp(t)}{\sqrt{t^2-x^2}}dt - \gamma\left\{\int_0^1 q(t)dt - x\int_0^x \frac{q(t)}{\sqrt{x^2-t^2}}dt\right\} = 1, \\ \displaystyle\gamma\int_0^x \frac{tp(t)}{\sqrt{x^2-t^2}}dt - x\int_x^1 \frac{q(t)}{\sqrt{t^2-x^2}}dt = 0. \end{cases} \tag{2.3}$$

The existence and uniqueness results of the system (2.3) on $[0, c)$ can be obtained from the existence of the solution of (2.1) on $[0, 1]$ which is given in [11]. Now, on the interval $[c, 1]$, the system of integral equations (2.1) together with (2.2) can be reduced to a single equation in terms of the normal stress $p(t)$ as follows.

$$\int_x^1 \frac{t-\gamma\mu\sqrt{t^2-x^2}}{\sqrt{t^2-x^2}}p(t)dt + \gamma\mu\int_0^x \frac{x-\sqrt{x^2-t^2}}{\sqrt{x^2-t^2}}p(t)dt = 1. \tag{2.4}$$

Substitution of the solution $p(x)$ gives an equation for the corresponding shear stress $q(x)$. Finally, the normal compliance of the surface can be written in terms of a dimensionless compliance factor

$$p^* = \frac{\pi}{2}\int_0^1 xp(x)dx. \tag{2.5}$$

Note that without having to solve the system completely, we only need to compute the normal stress p which provides an expression for the normal compliance of the surface, and is of our special interest.

2.2. The λ-matrix equation

The Abel integral equations of the Hertz contact problem (with finite friction) can be transformed to a λ-matrix equation through an appropriate choice of the variable. The variables of the multivariable λ-matrix will be chosen according to the integral operators of the system (2.3):

$$\begin{cases} \displaystyle(\kappa_1 f)(x) = \int_x^1 \frac{t}{\sqrt{t^2-x^2}}f(t)dt, \\ \displaystyle(\kappa_2 f)(x) = -\gamma\int_0^1 f(t)dt + \gamma\int_0^x \frac{x}{\sqrt{x^2-t^2}}f(t)dt, \\ \displaystyle(\kappa_3 f)(x) = \gamma\int_0^x \frac{t}{\sqrt{x^2-t^2}}f(t)dt, \\ \displaystyle(\kappa_4 f)(x) = -\int_x^1 \frac{x}{\sqrt{t^2-x^2}}f(t)dt. \end{cases} \tag{2.6}$$

We can therefore define a two-dimensional λ-matrix corresponding to the system as follows.

$$\mathbf{A}(\lambda) = \begin{pmatrix} \kappa_1 & \kappa_2 \\ \kappa_3 & \kappa_4 \end{pmatrix}, \tag{2.7}$$

where λ stands for the variables $(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$. In this formulation, the system can be written in terms of a single λ-matrix equation $\mathbf{A}(\lambda)\mathbf{X} = \mathbf{G}$, where $\mathbf{A}(\lambda)$ is shown in (2.7), $\mathbf{G} = [1, 0]^T$ is the constant vector on the right hand side of the equation and $\mathbf{X} = [p, q]^T$ is the unknown vector to be determined.

3. The λ-matrix approach

In this section, we develop an alternative integral formulation which allows us to construct a single integral equation by using the decomposition form of the λ-matrix. Before we discuss this issue further, we give some preliminaries regarding the λ-matrix concepts from [13,14].

Let $\mathcal{F}$ be a field, λ an indeterminate and $\mathcal{F}[\lambda]$ the polynomial ring over $\mathcal{F}$. A matrix, whose elements are polynomials of λ, is called matrix polynomial or λ-matrix. Let $M_{m,n}(\mathcal{F}[\lambda])$ be the set of all $m \times n$ λ-matrices and $GL_k(\mathcal{F}[\lambda])$ the set of all $k \times k$ invertible λ-matrices. We now review some well-known results from matrix theory that we need here.

While attempting to derive a numerical method, we should assume the integral operators as the variables of the λ-matrix to be commutative. The characterization of the λ-matrices can be extended to the multivariable λ-matrices in which the principal variable is considered as the variable that emerges in practice among all of the variables. Now, using canonical form, we will show that under certain suitable conditions, the desired solution of the λ-matrix equation can be obtained explicitly. In fact, a λ-matrix can be transformed into the canonical diagonal form by elementary transformations i.e. for any $\mathbf{A}(\lambda) \in M_{m,n}(\mathcal{F}[\lambda])$ with rank r, there exist $\mathbf{U}(\lambda) \in GL_m(\mathcal{F}[\lambda])$, $\mathbf{V}(\lambda) \in GL_n(\mathcal{F}[\lambda])$, and r polynomials $f_1(\lambda), \dots, f_r(\lambda)$ such that

$$\mathbf{D}_{m,n}(f_1(\lambda), \dots, f_r(\lambda), 0, \dots, 0) = \mathbf{U}(\lambda)\mathbf{A}(\lambda)\mathbf{V}(\lambda),$$

where $f_j(\lambda)|f_{j+1}(\lambda)$ for $j = 1, \dots, r-1$ and $f_j(0) \neq 0$ for $j = 1, \dots, r$ (see [23,13] for further details).

Let us consider the λ-matrix equation in $M_{2,2}(\mathcal{F}[\lambda])$ of the form:

$$\mathbf{A}(\lambda)\mathbf{X} = \mathbf{G},$$

where $\mathbf{A}(\lambda) \in M_{2,2}(\mathcal{F}[\lambda])$ is a λ-matrix, $\mathbf{G}$ is a given vector function and $\mathbf{X}$ is the unknown vector in a two-dimensional real vector space to be determined. The solution of the operator system is defined explicitly by $\mathbf{X} = \mathbf{V}(\lambda)\mathbf{Y}$, where

$$\mathbf{D}(\lambda)\mathbf{Y} = \mathbf{U}(\lambda)\mathbf{G}. \tag{3.1}$$

This procedure can be carried out to give the structure of a successful direct method for a coupled system of Volterra integral equations (2.3). We first mention the transformation of the λ-matrix to the canonical form, then applying the described method, we obtain an equivalent single integral equation for the normal stress.

In our considered problem, under the assumption of the commutativity of the variables, we get the decomposition form of the λ-matrix (2.7) in the following form:

$$\mathbf{U}(\lambda) = \begin{pmatrix} 0 & 1 \\ \kappa_4 & -\kappa_2 \end{pmatrix}, \qquad \mathbf{D}(\lambda) = \begin{pmatrix} \kappa_4^2 & 0 \\ 0 & (\kappa_4\kappa_1 - \kappa_2\kappa_3)\kappa_4 \end{pmatrix}, \qquad \mathbf{V}(\lambda) = \begin{pmatrix} 0 & \kappa_4 \\ \kappa_4 & -\kappa_3 \end{pmatrix}. \tag{3.2}$$

The second equation of the system (3.1) is in the form

$$[\kappa_4\kappa_1 - \kappa_2\kappa_3]\kappa_4 y_2 = \kappa_4 g_1 - \kappa_2 g_2.$$

Replacing the first equation of $\mathbf{X} = \mathbf{V}(\lambda)\mathbf{Y}$ as $x_1 = \kappa_4 y_2$, we can make the following operator equation:

$$[\kappa_4\kappa_1 - \kappa_2\kappa_3]x_1 = \kappa_4 g_1 - \kappa_2 g_2,$$

and after some manipulation involving combination of the operators defined in (2.6), the normal stress can be computed from the following integral equation:

$$\begin{aligned} &\int_x^1 \int_t^1 \frac{-xs}{\sqrt{(t^2-x^2)(s^2-t^2)}} p(s)dsdt + \gamma^2 \int_0^1 \int_0^t \frac{s}{\sqrt{t^2-s^2}} p(s)dsdt \\ &- \gamma^2 \int_0^x \int_0^t \frac{-xs}{\sqrt{(t^2-x^2)(s^2-t^2)}} p(s)dsdt = x\,Ln(x) - x\,Ln(1+\sqrt{1-x^2}). \end{aligned} \tag{3.3}$$

The method deals with transformation of a coupled Volterra integral equations to an equivalent two dimensional reduced equation. An inspection of the above numerical approach shows that it can provide reasonable results for the Hertz problem. We should emphasize that the derived Eqs. (2.4) and (3.3) contain one variable, and solving these equations can be considered superior to the other methods for obtaining the solution of the system of integral equations (2.1) directly.

The following algorithm summarizes the proposed λ-matrix approach for the Hertz contact problem:

Algorithm: The construction of the normal stress of the Hertz contact problem (2.1) together with the complementarity conditions (2.2) on $[0, 1]$

Considering the parameter c as the separating point of slippage, subdivide the problem in two cases:
Case I. On the interval $[c, 1]$, set $q(x) = \mu p(x)$, and compute the solution of (2.4).
Case II. On the interval $[0, c)$, set $u(x) = 0$, and consider the following steps to compute the solution of (2.3):
Step 1. Choose the integral operators (2.6) as the variable of the λ-matrix (2.7).
Step 2. Compute the Smith form of the λ-matrix (2.7) as the decomposition (3.2).
Step 3. Solve the reduced system $\mathbf{D}(\lambda)\mathbf{Y} = \mathbf{U}(\lambda)\mathbf{G}$ for the vector $\mathbf{Y}$.
Step 4. Compute the unknowns from the equation $\mathbf{X} = \mathbf{V}(\lambda)\mathbf{Y}$ which leads to the single equation (3.3) for the normal stress $p(t)$.
Step 5. Compute the dimensionless compliance factor (2.5).

For implicit schemes, a system of coupled integral equations is required to be solved. However, by using the canonical form, which eliminates one of the unknown functions in the system, it is straightforward to construct schemes for solving single equations with higher order.

Table 1
The absolute errors of the compliance factor for the known parameters corresponding to $\nu = 0$ and 0.25.

μ	i	$\nu = 0$		$\nu = 0.25$	
		Parameters (c_i)	Absolute errors (e_i)	Parameters (c_i)	Absolute errors (e_i)
	1	–	–	0.3000	1.88015E−7
0.1387	2	0.0750	6.80006E−8	0.3083	1.20242E−7
	3	0.0913	1.69129E−7	0.3028	3.23468E−7
	1	–	–	0.5000	2.48876E−7
0.1801	2	0.1750	1.04935E−7	0.5083	1.00765E−6
	3	0.2041	1.37615E−7	0.5083	1.00765E−6
	1	0.2400	2.2002E−7	–	–
0.2063	2	0.2417	1.59593E−7	0.6083	5.12846E−7
	3	0.2739	1.76297E−7	0.6124	1.11911E−6
	1	–	–	0.8000	1.20263E−6
0.2843	2	0.4750	4.58949E−7	0.8083	1.60851E−6
	3	0.4743	5.65967E−7	0.8010	4.39039E−8
	1	0.5000	4.34106E−7	–	–
0.2986	2	0.5083	5.07090E−7	0.8250	3.24977E−7
	3	0.5083	5.07090E−7	0.8317	1.72124E−6
	1	0.7000	8.56907E−7	–	–
0.4013	2	0.7083	8.90841E−7	0.9250	7.51138E−7
	3	0.7072	7.85729E−7	0.9265	2.60736E−6
	1	0.8000	2.49809E−6	–	–
0.4862	2	0.8083	1.83370E−6	0.9750	2.27621E−7
	3	0.8010	1.88343E−7	0.9618	7.13191E−7

Table 2
The absolute errors of the compliance factor for different parameters c and μ.

ν	Absolute errors		
	c 0.24, μ 0.2063	c 0.5, μ 0.2986	c 0.7, μ 0.4013
0	2.03110E−7	5.43533E−7	1.25759E−6
0.125	1.47628E−7	3.14373E−7	1.25759E−6
0.25	8.61150E−8	5.06974E−7	4.22542E−7
0.375	2.21204E−7	7.42386E−8	3.91238E−8

4. Numerical experiments and some comments

The described algorithm considers the Hertz contact problem under the assumption of Coulomb friction with coefficient μ in the region of contact. Then, it computes the normal stress, as well as the compliance factor of an elastic interaction.

We work with the same conditions as [11,12]. Our numerical experiments show that the normal stress calculated by the λ-matrix approach gives a relatively satisfactory error when compared with the exact Hertzian model at long simulation times. Clearly, our proposed method considerably simplifies the computation of the normal stress compared with the given algorithm in [12], while the single equations can be solved by any spectral method.

In the proposed algorithm, the normal contact force includes the profile of the normal stress p and the tangential shear stress q computed by substituting the normal stress in the system. To validate the method of this paper, a comparison for the compliance factor is made between the λ-matrix approach and the exact solutions given in [11]. Moreover, the coefficient c, the point where adhesion terminates and slippage commences, can have various values for different cases of contact. Its values should be calculated from numerical or, even better, physical experiments.

Concerning the computational efficiency of the method, we note that the algorithm has been compared for a limited range of the parameters ν, μ and c given in [12,11]. We should mention that in Table 1, c_1 represents the limited values of c obtained by Spence [11], and c_2, c_3 refers to the values of c obtained by Gauthier et al. [12] with transformed and untransformed variables, respectively.

The behavior of the errors in Tables 1 and 2 can be explained through the fact that the solutions of the single equation are more accurate than the solutions of the system. We conclude that starting division of the interval [0, 1] by the parameter c togethers with an appropriate method for applying the λ-matrix properties appears to be a useful tool for the numerical computation of the solution of the integral equation systems.

In Table 2, an accuracy test is also provided by the case of the compliance factor, for which μ and c can be obtained from the results of [11], and the calculation was carried out for $\nu = 0, 0.125, 0.25$ and 0.375.

The numerical experiments show that the compliance factor of the compared vectors coincide to a very high degree of accuracy over the entire simulation period. The maximum errors confirm the effectiveness of the proposed method.

5. Concluding remarks

The integral formulation of the Hertz contact problem with finite friction has been completely characterized in this paper. This problem was formulated as a system of integral equations, and the proposed algorithm was used to convert it to a λ-matrix equation. Thereby providing a matrix polynomial to the problem, as well as its Smith factorization gives an efficient approach for finding its solution. Actually, the referred matrix polynomial transforms the system to a single integral equation which yields the compliance factor. In future study, we will discuss rules and heuristics for choosing the general integral operator that should be used to determine a λ-matrix corresponding to the system. It seems clear that more experiments are necessary to establish guidelines, particularly when there are several choices of integral operators. The extension of the results for system of nonlinear integral equations, which happens more in natural phenomena, could be also the purpose of the future work.

References

[1] Y. Gonthier, C. Lange, J. McPhee, On implementing a bristle friction model in a contact model based on volumetric properties, in: Proc. ECCOMAS Thematic Conference on Multibody Dynamics, Milan, Italy, June 25–28, 2007.
[2] D.A. Spence, Self similar solutions to adhesive contact problems with incremental loading, Proc. R. Soc. Lond. Ser. A 305 (1968) 55–80.
[3] B. Noble, D.A. Spence, Formulation of two dimensional and axisymmetric problems for an elastic half-space, Technical Report 1089, Mathematics Research Center, University of Wisconsin, 1971.
[4] R.W.G. Cottle, F.L. Giannessi, J.-L. Lions (Eds.), Variational Inequalities and Complementarity Problems, Wiley, New York, 1980.
[5] M.D. Bryant, L.M. Keer, Rough contact between elastically and geometrically identical curved bodies, J. Appl. Mech. 49 (1982) 345–352.
[6] A. Sackfield, D.A. Hills, Some useful results in the classical Hertz contact problem, J. Strain Anal. 18 (1983) 101–105.
[7] A. Sackfield, D.A. Hills, Some useful results in the tangentially loaded Hertzian contact problem, J. Strain Anal. 18 (1983) 107–110.
[8] N. Kikuchi, J.T. Oden, Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods, SIAM, Philadelphia, PA, 1988.
[9] L. Johansson, A. Klarbring, The rigid punch problem with friction using variational inequalities and linear complementarity, Mech. Struct. Mach. 20 (3) (1992) 293–319.
[10] M. Anitescu, J.F. Cremer, F.P. Potra, Properties of complementarity formulations for contact problems with friction, Technical Report 83, Department of Mathematics, The University of Iowa, 1995.
[11] D.A. Spence, The Hertz contact problem with finite friction, J. Elasticity 5 (1975) 297–319.
[12] A. Gauthier, P.A. Knight, S. McKee, The Hertz contact problem, coupled Volterra integral equations and a linear complementarity problem, J. Comput. Appl. Math. 206 (2007) 322–340.
[13] I. Gohberg, P. Lancaster, L. Rodman, Matrix Polynomials, Academic Press, New York, 1982.
[14] P. Lancaster, M. Tismenetsky, The Theory of Matrices, second ed., Academic Press, Orlando, 1985.
[15] M.V. Bulatov, Application of matrix polynomials to investigation of singular equations, Lobachevskii J. Math. 20 (2005) 17–29.
[16] A. Amiraslani, The equivalence of the constrained Rayleigh quotient and Newton methods for matrix polynomials expressed in different polynomial bases along with the confluent case, Int. J. Comput. Math. (2011) 1–23.
[17] J.A. Foster, Algorithms and techniques for polynomial matrix decompositions, Ph.D. Thesis, Cardiff University, 2008.
[18] M.K.H. Fan, A.L. Tits, *m*-form numerical range and the computation of the structured singular value, IEEE Trans. Automat. Control 33 (1988) 284–289.
[19] T. Kailath, Linear Systems, Prentice Hall, Englewood Cliffs, 1980.
[20] P. Van Dooren, P. Dewilde, The eigenstructure of an arbitrary polynomial matrix: computational aspects, Linear Algebra Appl. 50 (1983) 545–579.
[21] H.H. Rosenbrock, State-Space and Multivariable Theory, Wiley, New York, 1970.
[22] I.I. Argatov, The solution of the Hertz axisymmetric contact problem, J. Appl. Math. Mech. 70 (2006) 621–635.
[23] J. Wilkening, J. Yu, A local construction of the Smith normal form of a matrix polynomial, J. Symbolic Comput. 46 (2011) 1–22.