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Some Explicit Class of Hybrid Methods for Optimal Control of Volterra Integral Equations

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Abstract. In this paper, we first extend and analyze the steepest descent method for solving optimal control problem for systems governed by Volterra integral equations. Then, we present some hybrid methods based on the extended steepest descent and two-step Newton methods, to solve the problem. The global convergence results are also established using some mild assumptions and conditions. Numerical results show that the proposed hybrid methods are more powerful and faster than the extended steepest descent method.

Keywords: Optimal control, Steepest descent, Two-step Newton's method, Volterra integral equation.

1. Introduction

The classical theory of optimal control was originally developed to deal with systems of controlled ordinary differential equations [9]. A wide class of control systems can be described by Volterra integral equations instead of ordinary differential equations. It is well-known that Volterra integral equations can be used to model many classes of phenomena, such as population dynamics, continuum mechanics of materials with memory economic problems, the spread of epidemics, non-local problems of diffusion and heat conduction problem. An excellent survey on applications of Volterra integral equation can be found in [6] and [8] .

The problem of optimal control of systems governed by Volterra integral equations has been studied by many authors e.g. Vinokurov [10], Medhin [11], Schmidt [13] and Belbas [3, 4, 5]. The methods that are usually employed for solving this problem are based on the necessary conditions obtained using Pontryagin's maximum principle. Belbas in [3] presented a method based on discretization of the original Volterra integral equation and a novel type of dynamic programming in which the Hamilton-Jacobi function is parametrized by control function. In more recent work of Belbas [4], the controlled Volterra integral equations are approximated by a sequence of controlled ordinary differential equations and the resulting approximating problems can then be solved by dynamics programming methods for ODEs controlled systems. The interested reader may found some references on optimal control of Volterra integral equations by methods other than dynamic programming in [5,7, 10, 11].

Due to the difficulties in obtaining analytical solution for the problem, the numerical methods have been usually interested. Belbas in [5] described several iterative schemes for solving Volterra optimal problems and analyzed the conditions that guarantee the convergence of the methods. Schmidt in [14], proposed some direct and indirect numerical methods for solving optimal control problems governed by ODEs as well as integral equations.

In this paper, we are going to provide some explicit iterative methods based on the necessary conditions for solving Volterra optimal control problems in which the control variables are not constrained by any boundaries. We first generalized the Steepest Descent (**SD**) method for solving the problem and then hybridize the SD and Two-Step Newton (**TSN**) methods in order to efficiently solving the Volterra optimal control problem. The proposed hybrid method integrates the SD and TSN methods to obtain global convergence results together with fast convergence rate. Our numerical results show that the hybrid schemes are more powerful and faster than the SD method.

The paper is organized as follows: the Volterra optimal control problem and some elementary related results are stated in section 2. Section 3 is devoted to describe and analyze the generalized SD and TSN methods. The new hybrid schemes based on SD and TSN methods are also constructed in this section. The global convergence of the proposed hybrid methods is analyzed in section 4 and finally some numerical results are given in section 5 to show the efficiency of the proposed hybrid methods in comparison with the SD method.

2. Problem Statement

Consider a controlled Volterra integral equation of the form

$$x(t) = x(a) + \int_a^t f\big(t,s,x(s),u(s)\big)\, ds, \tag{1}$$

where the continuous real-valued functions $x(t)$ and $u(t)$ are the state of the controlled system and the control function, respectively. It is assumed that the state and control variables are not constrained by any boundaries. In this paper, we consider the optimal control problem in which the cost functional J defined by

$$J = \int_a^b F\big(t,x(t),u(t)\big)dt, \tag{2}$$

is minimized under the Volterra integral equation given by (1). We are looking for the vector (x^*,u^*) which solves the following problem

$$\begin{aligned} &max && J_1 = -J, \\ &s.t. && x(t) = x(a) + \int_a^t f\big(t,s,x(s),u(s)\big)ds. \end{aligned} \tag{3}$$

It is well known that the Pontryagin maximum principle, or simply maximum principle, gives the necessary conditions for the optimal vector (x^*,u^*) of the problem (3), so through out the paper we assumed that the following conditions are satisfied:

$\boldsymbol{H_1}$: The conditions that guarantee the existence of a unique continuous solution to the integral equation (4), which include continuity of the function f for all s,t with $s \le t$, together with its Lipschitz condition.

$\boldsymbol{H_2}$: The partial derivatives f_x and f_u exist and are continuous, and for all t,s with $t \le s$, $f(t,s,x,u) = 0$.

$\boldsymbol{H_3}$: $F(t,x,u)$ is a smooth function.

Following [15], using these assumptions, the Pontryagin maximum principle can be stated as follows : Suppose that the function $u^*(t)$, $a \le t \le b$, and related state function $x^*(t)$, solve the problem (3). Then, there exist a continuous multiplier function $\lambda^*(t)$, and a Hamiltonian $H(t,x,u,\lambda)$, defined by

$$H\big(t,x(t),u(t),\lambda(t)\big) = F\big(t,x(t),u(t)\big) - \int_t^b f\big(s,t,x(t),u(t)\big)\lambda(s)ds,$$

such that, for each t,

1. $u^*(t)$ minimize $H(t,x,u,\lambda)$, i.e. $\frac{\partial H}{\partial u}|_{u=u^*(t)} = 0$, providing that H is differentiable

   with respect to u.

2. $\lambda^*(t) = -\frac{\partial H}{\partial u}|_{u=u^*(t),x=x^*(t)}$.

3. Iterative Methods

In this section, using maximum principle, we present some iterative methods for solving (3). First of all, we state and analyze an extension of the steepest descent method for solving optimal control of Volterra integral equations. We then introduce the Newton and its variety, the so-called two-step Newton methods and related convergence results. Finally, a combination of these methods are provided as a new hybrid method for solving the optimal control of Volterra integral equation.

3.1. Steepest Descent Method

Here, we briefly describe an iteration of the steepest descent method when we apply to solve problem (3). Let $u^{(i)}$ be the control function in the i^{th}-iteration of the steepest descent method, then the state function $x^{(i)}(t)$, in the i^{th}-iteration is given by

$$x^{(i)} = x(a) + \int_a^t f\left(t,s,x^{(i)}(s),u^{(i)}(s)\right)ds. \tag{4}$$

Using maximum principle, the related multiplier $\lambda^{(i)}(t)$, is then provided by

$$\lambda^{(i)}(t) = -\frac{\partial F}{\partial x}\left(t,x^{(i)}(t),u^{(i)}(t)\right) + \int_t^b \frac{\partial f}{\partial x}\left(s,t,x^{(i)}(t),u^{(i)}(t)\right)\lambda^{(i)}(s)ds. \tag{5}$$

Let us set

$$\frac{\partial H^{(i)}}{\partial u} = \frac{\partial H}{\partial u}\left(t,x^{(i)}(t),u^{(i)}(t),\lambda^{(i)}(t)\right). \tag{6}$$

For a given $\epsilon > 0$, $(x^{(i)}(t),u^{(i)}(t))$ is an ε-solution of the problem (3), if

$$\|\frac{\partial H^{(i)}}{\partial u}\|^2 < \varepsilon \tag{7}$$

where $\|.\|$ is the continuous 2-norm defined by

$$\|\frac{\partial H^{(i)}}{\partial u}\|^2 \triangleq \int_a^b [\frac{\partial H^{(i)}(t)}{\partial u}]^2 dt. \tag{8}$$

Otherwise, the new iteration for $u^{(i+1)}(t)$ is computed by

$$u^{(i+1)}(t) = u^{(i)}(t) - \tau \frac{\partial H^{(i)}}{\partial u}, \quad (9)$$

where $\tau > 0$ is the step size which is determined by exact line search rule, i.e.,

$$\tau = arg_{\alpha > 0}^{min} \frac{\partial H}{\partial u}\left(t, x^{(i)}(t), u^{(i)}(t) - \alpha \frac{\partial H^{(i)}}{\partial u}(t), \lambda^{(i)}(t)\right). \quad (10)$$

The above procedure is repeated by replacing $u^{(i)} \leftarrow u^{(i+1)}$ until the stopping criteria (7) is satisfied.

3.2 Newton Method

Newton method is one of the most attractive iterative method for approximating the solution of a nonlinear operator system. Let G be a continuous and Frechet differentiable operator defined on an open convex subset D of a Banach space X and x^* be a unique solution of the system

$$G(x) = 0,$$

then, under some assumptions indicated in the Theorem 1, the Newton iterative scheme

$$u_{n+1} = u_n - G'(u_n)^{-1} G(u_n), \qquad (n \geq 0) \quad (11)$$

is well-defined and converges to the unique solution x^* with quadratic convergence rate. Now, we recall the following convergence theorem for the Newton method based on the well-known Newton-Kantorovich hypothesis which its proof can be found in [17]:

Theorem 1. (From [1]) *Let X and Y are Banach spaces and $G: X \longrightarrow Y$ be continuously Frechet differentiable on some open convex subset $D \subset X$. Assume that:*

a) *there exist a point $x_0 \subset D$\$, such that*
$$G'(x_0)^{-1} \subset L(Y, X),$$
where $L(Y, X)$ is a set of all bounded linear operators from Y to X.

b) *there exist a Lipschitz constant $L > 0$, such that for all $x, y \in D$,*
$$\| G'(x_0)^{-1}[G'(x) - G'(y)] \| \leq L \| x - y \|.$$

c) *for some positive constant η,*
$$h_L = 2\eta L \leq 1,$$
and $U(x_0, r^*) \subset D$, where
$$r^* = \frac{2\eta}{1 + \sqrt{1 - h_L}}.$$

Then, the Newton iterates $\{x_n\}_{n \geq 0}$ are well defined, remain in $U(x_0, r^*)$ and converge to the unique zero x^* of $F(x) = 0$ in $U(x_0, r^*)$.

Two modifications of the Newton method are two-step Newton methods with order of convergence 3 and 4. Let $u_0 \subset D$ and an scalar $\gamma \geq 0$ be given, the two-step Newton method with order of convergence 3 generates the iterations based on the following scheme (See e.g. [2]):

$$y_n = u_n - (G'(u_n)^{-1})\, G(u_n), \qquad (n \geq 0)(u_0 \in D)$$

$$u_{n+1} = y_n - (G'(u_n)^{-1})G(y_n). \tag{12}$$

This method needs one inverse and two function evaluation at every step. In [2], it is proven that if the Newton-Kantorovich hypothesis holds, then this method converges to the unique solution of the system with cubic convergence rate. Also, following[2] , the two-step Newton method with order of convergence 4, generates the iterations based on the following scheme:

$$y_n = u_n - (G'(u_n)^{-1})\, G(u_n), \qquad (n \geq 0)\ (u_0 \in D)$$

$$u_{n+1} = y_n - (G'(y_n)^{-1})G(y_n). \tag{13}$$

Clearly, the computational complexity of this method consist of two inverses and two function evaluations at every step and the convergence of the proposed method is also guaranteed using the Newton-Kantorovich hypothesis with rate of convergence 4. Note that the stopping condition for both methods is $|G(u_n)| \leq \gamma$.

3.3 Hybrid Methods

In this subsection, we hybridize the steepest descent method with three kind of Newton methods, provided in subsection 3.2, in order to solve the optimal control of Volterra integral equation (3). We note that the steepest descent method is globally convergent while the Newton methods are not. Therefore, these hybridizations lead us to have globally convergent methods with fast convergence rate compared to the steepest descent method. In a short view, the new schemes start and continue the iterates by the SD methods while the Newton Kantorovich hypothesis are not satisfied. Otherwise, we follow the iterations by TSN methods. The following algorithm summarizes our strategy:

Algorithm 1.

Step 1: Given the precise parameter $\gamma > 0$, choose an arbitrary control function $u^{(0)}(t), t \in [a, b]$, and set $i = 0$.

Step 2: Solve the integral equation (4), for $x^{(i)}(t), \quad t \in [a, b]$.

Step 3: Solve the integral equation (5), for $\lambda^{(i)}(t), \quad t \in [a, b]$.

Step 4: Calculate $\frac{\partial H^{(i)}(t)}{\partial u}$, defined by (6).

Step 5: If $\| \frac{\partial H^{(i)}(t)}{\partial u} \|^2 \leq \gamma$, then stop. ($x^{(i)}$ and $u^{(i)}$ satisfies the necessary conditions)

otherwise, if Newton-Kantorovich hypothesizes hold, then generate a new control function given by

$$u^{(i+1)} = P\left(u^{(i)}, y^{(i)}\right), \tag{14}$$

else, set

#

$$u^{(i+1)} = u^{(i)} - \tau \left(\frac{\partial H}{\partial u}\right)^{(i)},$$

where τ is given by (10).

Step 6: Set $i \leftarrow i + 1$ and go to step 2.

Remark. Note that in the step 5 of the proposed algorithm, the operator P is chosen appropriately according to the Newton's method that we would like to use, more precisely:

1. If we use the classic Newton's method,

$$P = u^{(i)} - \left(\frac{\partial^2 H^{(i)}}{\partial u^2}\right)^{-1} \left(\frac{\partial H^{(i)}}{\partial u}\right).$$

2. If we use the two-step Newton method (12),

$$P\left(u^{(i)}, y^{(i)}\right) = y^{(i)} - \left(\frac{\partial^2 H^{(i)}}{\partial u^2}\big|_{u=y^{(i)}}\right)^{-1} \left(\frac{\partial H^{(i)}}{\partial u}\big|_{u=y^{(i)}}\right), \tag{15}$$

where $y^{(i)} = u^{(i)} - \left(\frac{\partial^2 H^{(i)}}{\partial u^{(2)}}\right)^{-1} \left(\frac{\partial H^{(i)}}{\partial u}\right)$.

3. If we use the two-step Newton method (13),

$$P\left(u^{(i)}, y^{(i)}\right) = y^{(i)} - \left(\frac{\partial^2 H^{(i)}}{\partial u^2}\big|_{u=u^{(i)}}\right)^{-1} \left(\frac{\partial H^{(i)}}{\partial u}\big|_{u=y^{(i)}}\right), \tag{16}$$

where $y^{(i)} = u^{(i)} - \left(\frac{\partial^2 H^{(i)}}{\partial u^2}\right)^{-1} \left(\frac{\partial H^{(i)}}{\partial u}\right)$.

4. Convergence Analysis

In this section, we first provide the convergence result for the steepest descent method. To do so, we assume that the conditions $H_1 - H_3$ together with the following conditions are satisfied:

$\boldsymbol{H_4}$: The partial derivatives f_x and F_x exist and are Lipschitz continuous.

$\boldsymbol{H_5}$: The conditions that guarantee existence and uniqueness of optimal control be satisfied. (See e.g. [19,18]).

The following theorem provides the convergence result for the steepest descent method which is generalized to the Volterra optimal control problem. Some part of the proof is a repetition of the standard proof of an extremum principle of Pontryagin's type for Volterra integral equations which exists in the [15].

Theorem 2. Suppose that the assumptions $H_1 - H_5$ are satisfied. Then, the steepest descent method described in section 3.1 is convergent to the optimal solution of the problem (3).

Proof: Let u be an arbitrary control function. For all continuous functions $\Delta u(t)$, the variation of J_1 is given by

$$\delta J_1(u, \Delta u)=\lim_{\varepsilon\to 0}\frac{J_1(u+\varepsilon\Delta u)-J_1(u)}{\varepsilon},$$

where ε is an arbitrary small number. The computation of this variation yields:

$$\delta J_1(u, \Delta u)=\int_a^b\left[-F_x(t,x(t,u),u)\frac{\partial}{\partial\varepsilon}x(t,u+\varepsilon\Delta u)|_{\varepsilon=0}-F_u(t,x(t,u),u)\Delta u\right]dt \qquad (17)$$

Now from (1), we have

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}x(t,u+\varepsilon\Delta u)|_{\varepsilon=0}\\ &=\int_a^t\left[f_x(t,s,x(s,u),u)\frac{\partial}{\partial\varepsilon}x(s,u+\varepsilon\Delta u)|_{\varepsilon=0}\right.\\ &\left.+f_u(t,s,x(t,u),u)\Delta u(s)\right]ds. \qquad (18)\end{aligned}$$

Let us assume that

$$A(t,s)=f_x\big(t,s,x(s,u),u(s)\big),$$

$$B(t,s)=f_u\big(t,s,x(s,u),u(s)\big),$$

$$y(t)=\frac{\partial}{\partial\varepsilon}x(t,u+\varepsilon\Delta u)|_{\varepsilon=0}. \qquad (19)$$

Substituting (19) into (18), leads us to the following integral equation:

$$y(t)=\int_a^t[A(t,s)y(s)+B(t,s)\Delta u(s)]ds. \qquad (20)$$

The solution of (20) can be obtained as:

$$y(t)=\int_a^t r(t,s)\left[\int_a^s B(s,\tau)\Delta u(\tau)d\tau\right]ds+\int_a^t B(t,s)\Delta u(s)ds, \qquad (21)$$

where the resolvent kernel $r(t,s)$ is defined by:

$$r(t,s)=A(t,s)+\int_s^t r(t,\tau)A(\tau,s)d\tau.$$

One knows that, for a given continuous function g on the triangular region $a\le\tau\le t\le T$, we have:

$$\int_a^T \int_a^t g(t,\tau)d\tau dt = \int_a^T \int_t^T g(t,\tau)d\tau dt. \tag{22}$$

Rewriting (21) by (22), gives rise:

$$y(t) = \int_a^t \left[B(t,s) + \int_s^t r(t,\tau)B(\tau,s)d\tau\right]\Delta u(s)ds. \tag{23}$$

Recalling (19) and substituting (23) into (17) implies that

$$\delta J_1(u,\Delta u) = \int_a^b \left\{-F_x(t,x(t,u),u(t))\left[\int_a^t \left[B(t,s) + \int_s^t r(t,\tau)B(\tau,s)d\tau\right]\Delta u(s)ds\right] - F_u(t,x(t,u),u(t))\right\}\Delta u(t)dt. \tag{24}$$

Now, applying (22) to (24) yields:

$$\delta J_1(u,\Delta u) = \int_a^b \left\{\int_t^b -F_x(s,x(s,u),u(s))\left[B(t,s) + \int_t^s r(s,\tau)B(\tau,t)d\tau\right]ds - F_u(t,x(t,u),u(t))\right\}\Delta u(t)dt. \tag{25}$$

Let $H_1 = -H$, we will show that

$$\frac{\partial H_1}{\partial u} = \int_t^b -F_x(s,x(s,u),u(s))[B(t,s) + \int_t^s r(s,\tau)B(\tau,s)d\tau]ds - F_u(t,x(t,u),u(t))\Delta u(t)dt,$$

and therefore by evaluating (25) in $u = u^{(i)}$, we have

$$\delta J_1\left(u^{(i)},\Delta u^{(i)}(t)\right) = \int_a^b \frac{\partial H_1}{\partial u}^{(i)} \Delta u^{(i)}(t)dt = \int_a^b \left(-\frac{\partial H^{(i)}}{\partial u}\right)(-\tau\frac{\partial H^{(i)}}{\partial u})dt = \tau\int_a^b \left(\frac{\partial H^{(i)}}{\partial u}\right)^2 dt \geq 0,$$

which concludes that $\delta J^{(i)} \leq 0$. Therefore, J is decreased in each iteration which implies that the steepest descent method is convergent to optimal solution. Now, we show that (26) holds. Let us denote the right hand side of (26) by I. Using (22), this equation can be written as:

$$I = \int_t^b \left[-F_x(s,x(s,u),u(s)) - \int_s^b F_x(\tau,x(\tau,u),u(\tau))r(\tau,s)d\tau\right]B(s,t)ds - F_u(t,x(t,u),u(t)). \tag{27}$$

According to Theorem 1 in [15], we know that $\lambda(s)$, defined by

$$\lambda(s) = -F_x(s,x(s,u),u(s)) + \int_s^b -F_x(\tau,x(\tau,s),u(\tau))r(\tau,s)d\tau, \tag{28}$$

is consistent with

$$\lambda(s) = -F_x(s,x(s,u),u(s)) + \int_s^b f_x(t,x(t,u),u(t))\,\lambda(t)dt.$$

Thus, substituting (28) in (27) will lead us to the following equation:

$$\mathrm{I} = -\mathrm{F_u}(\mathrm{t},\mathrm{x}(\mathrm{t},\mathrm{u}),\mathrm{u}(\mathrm{t})) + \int_\mathrm{t}^\mathrm{b} \lambda(\mathrm{s})\mathrm{B}(\mathrm{s},\mathrm{t})\mathrm{ds}. \tag{29}$$

Now, applying (19) and (29), we obtain:

$$-F_u\left(t,x(t,u(t)),u(t)\right) + \int_t^b \lambda(s)f_u(s,x,u,t)ds = \frac{\partial H_1}{\partial u},$$

and this completes the proof of the theorem. ■

In this part, we establish the main theorem of the paper which is related to the convergence property of th sequence of control $\{u^{(i)}\}$, generated by Algorithm 1. It is well-known that $C([0,T],\mathbb{R})$ is a Banach spac and therefore Theorem 1 can be applied, so the following theorem states that the control sequenc $\{u^{(i)}\}$ generated by Algorithm 1 converges to the point in which the maximum principle is satisfied:

Theorem 3. *Suppose that the assumptions $H_1 - H_5$ hold. Let the operator $G = \frac{\partial H}{\partial u}$ satisfies the Newton-Kantrovich hypothesis which is stated in Theorem 1. Then, the sequences $\{u^{(i)}\}$, $\{x^{(i)}\}$ and $\{\lambda^{(i)}\}$, that are generated by Algorithm 1, are convergent to the point (u^*, x^*, λ^*) in which the maximum principle holds.*

Proof: Using condition H_5 and Theorem 2, we conclude that the steepest descent method is globall convergent to the unique control u^*. Thus, after n_1 iteration of steepest descent method, the control $u^{(n_1)}$ and therefore $x^{(n_1)}$ and $\lambda^{(n_1)}$, satisfies the Newton-Kantorvich hypothesis stated in Theorem 1. Hence, fron the n_1-th iteration on, we can use the Newton (or two-step Newton) method to get close to the unique contrc u^* and related x^* and λ^*, as desired and this completes the proof. ■

5. Numerical Experiments and Discussions

We illustrate the performance of the proposed hybrid methods and steepest descent method for Three test problems in order to compare the number of iterations and convergence rate that are required for reaching the accuracy of $\varepsilon = 10^{-8}$. In all of the test problems, the initial control has been set to zero. All the computations have been done with $Maple^{®}$ software. For ease of reference, we use the following notations in the Tables and Figures:

SD: Steepest Descent method
SDN: Hybridization of Steepest Descent and Newton method
SDN1: Hybridization of Steepest Descent and two-step Newton method stated in (13)
SDN2: Hybridization of Steepest Descent and two-step Newton method stated in (12)

Example 1. Consider the minimization of the functional

$$J = \int_0^1 tx(t) - u(t) + e^{2u(t)} dt,$$

subject to the integral equation

$$x(t) = \int_0^t tx(s) + tu(s) ds. \tag{30}$$

The Hamiltonian function can be stated as

$$H = tx(t) - u(t) + e^{2u(t)} - \int_t^1 \lambda(s)\big(sx(t) + su(t)\big) ds.$$

Let us suppose that u^* minimize H. Using the maximum principle, we express the necessary conditions for optimality as:

$$\lambda^*(t) = -\frac{\partial H}{\partial x} = -t + \int_t^1 s\lambda^*(s)ds, \tag{31}$$

$$2e^{2u^*(t)} - 1 - \int_t^1 s\lambda^*(s)ds = 0, \tag{32}$$

$$x^*(t) = \int_0^t tx^*(s) + tu^*(s)ds. \tag{33}$$

The analytical solution of the integral equation (31) may be obtained as:

$$\lambda^*(t) = t + e^{\frac{t^2}{2}}\left[e^{-\frac{1}{2}} - \frac{1}{2}\sqrt{2\pi}\,\mathrm{erf}\left(\frac{\sqrt{2}}{2}\right) - te^{\frac{-t^2}{2}} + \frac{1}{2}\sqrt{2\pi}\,\mathrm{erf}\left(\frac{\sqrt{2}}{2}t\right)\right],$$

where

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{t^2}dt.$$

So, from (32), we have

$$u^*(t) = \frac{1}{2}\ln\left[\frac{1}{2} + \frac{1}{2}\int_t^1 s\lambda^*(s)\,ds\right],$$

and finally the optimal state variable x^* can be obtained from the linear Volterra integral equation (33).

Table 1: Numerical result of Example 1 for SD and three hybrid methods

Itr	SD	SDN	SDN1	SDN2
1	0.185994612×10^1	0.185994612×10^1	0.185994612×10^1	0.185994612×10^1
2	0.292584772×10^0	0.292584772×10^0	0.292584772×10^0	0.292584772×10^0
3	$0.978499400 \times 10^{-1}$	0.292584772×10^0	$0.978499400 \times 10^{-1}$	$0.978499400 \times 10^{-1}$
4	$0.838472588 \times 10^{-2}$	$0.838472588 \times 10^{-2}$	$0.838472588 \times 10^{-2}$	$0.838472588 \times 10^{-2}$
5	$0.708728586 \times 10^{-2}$	$0.838472588 \times 10^{-2}$	$0.461994962 \times 10^{-5}$	$0.796714880 \times 10^{-3}$
6	$0.124292560 \times 10^{-3}$	$0.461993516 \times 10^{-5}$	$0.2118656096 \times 10^{-14}$	$0.601632398 \times 10^{-10}$
7	$0.692690569 \times 10^{-7}$	$0.270214306 \times 10^{-11}$		
8	$0.985570304 \times 10^{-11}$			

Table 2: The errors of the actual and numerical control for Example 1.

methods	Errors
SD	0.22149×10^{-5}
SDN	0.47973×10^{-8}
SDN1	0.10391×10^{-10}
SDN2	0.23091×10^{-5}

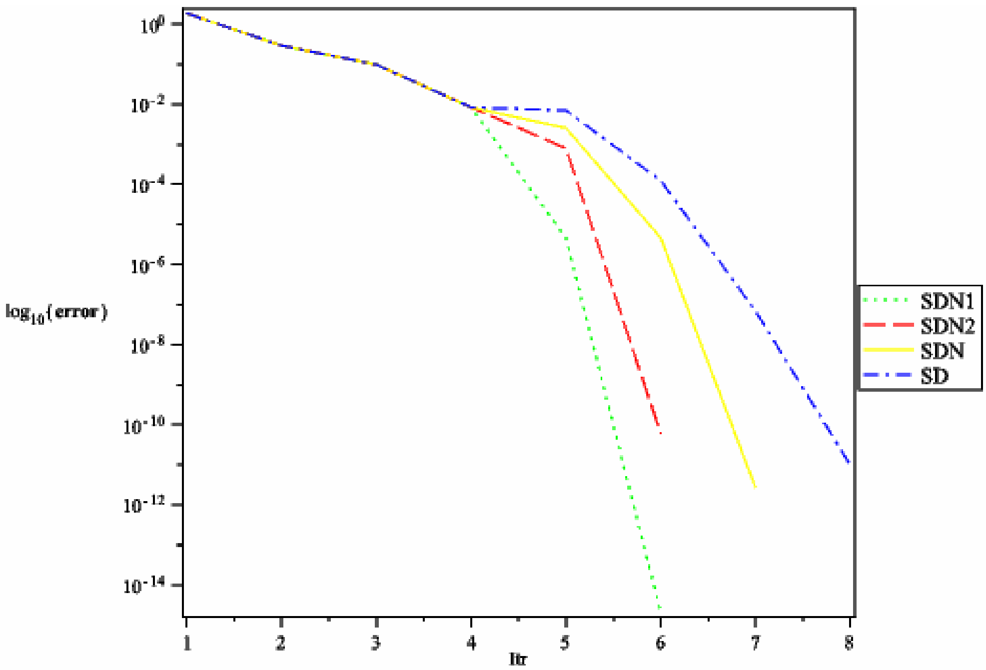

Fig. 1: The error behaviours of SD and three hybrid methods of Example 1.

Example 2.

$$\min J = \int_0^1 tx(t) - u^2(t) + e^{2u(t)}\,dt,$$

$$\text{s.t. } x(t) = \int_0^t tx(s) + tu(s)ds.$$

Example 3.

$$\min J = \int_0^1 tx(t) - u^4(t)dt,$$

$$\text{s.t. } x(t) = \int_0^t tx(s) + tu(s)ds.$$

Table 3: Numerical result of Example 2 for SD and three hybrid methods

Iter	SD	SDN	SDN1	SDN2
1	0.226720×10^{-1}	0.226720×10^{-1}	0.226720×10^{-1}	0.22672×10^{-1}
2	0.180115×10^{-1}	0.180115×10^{-1}	0.180115×10^{-1}	0.18011×10^{-1}
3	0.106793×10^{-1}	0.106793×10^{-1}	0.106793×10^{-1}	0.10679×10^{-1}
4	0.184614×10^{-2}	0.231536×10^{-3}	0.133046×10^{-7}	0.98134×10^{-5}
5	0.283223×10^{-3}	0.133046×10^{-7}	0.139000×10^{-19}	0.1199×10^{-15}
6	0.481042×10^{-4}			

In order to compare the number of iterations and convergence behaviour of the steepest descent and three hybrid methods, we solved the test problems for the same conditions. It turned out for given conditions, all of the methods produced almost the same accuracies in the first two or four iterations of the Algorithm. The number of iterations for reaching the given accuracy is also reported. In Tables 1, 3 and 4 we have compared the optimal control governed from the maximum principle with the numerical solutions which are obtained from the proposed hybrid methods. The value of $\| \frac{\partial H(\cdot)}{\partial u} \|^2$ produced at each iterations together with the corresponding iteration number of the algorithms are also listed in these Tables. The error behavior of the methods has also been shown in Figures 1-3. It is worth mentioning to indicate that the vertical axis in the figures provides the $\log_{10} error$ which in fact is $\log_{10}\left(\| \frac{\partial H(\cdot)}{\partial u} \|^2\right)$. Table 2 provides the errors between the exact solution u^* of the Example 1 and the numerical solution $\bar{u}$, which is obtained from the iterative methods, with the following formula:

$$\| \bar{u} - u^* \| = \int_a^b |\bar{u}(t) - u^*(t)| dt. \tag{34}$$

It is seen that among all of the proposed iterative methods, combination of steepest descent and two-step Newton methods has the best rate of convergence for reaching accuracy of 10^{-8}. However, considering SDN1 and SDN2, the first one gives the better convergence results.

Table 4.. Numerical results of Example 3 for SD and three hybrid methods.

Itr	SD	SDN	SDN1	SDN2
1	0.3454434×10^1	0.3454434×10^1	0.345443×10^1	0.3454434×10^1
2	0.2857226×10^1	0.2857226×10^1	0.285722×10^1	0.2857226×10^1
3	0.1879981×10^1	0.1736043×10^0	0.41128×10^{-3}	0.4342500×10^{-1}
4	0.1575350×10^1	0.411285×10^{-3}	0.12175×10^{-18}	0.9061416×10^{-7}
5	0.1147397×10^1	0.261778×10^{-8}	0.139000×10^{-19}	$0.1199313 \times 10^{-15}$
6	0.9719095×10^0	0.113420×10^{-18}		
7	0.7451515×10^0			
8	0.6355515×10^0			

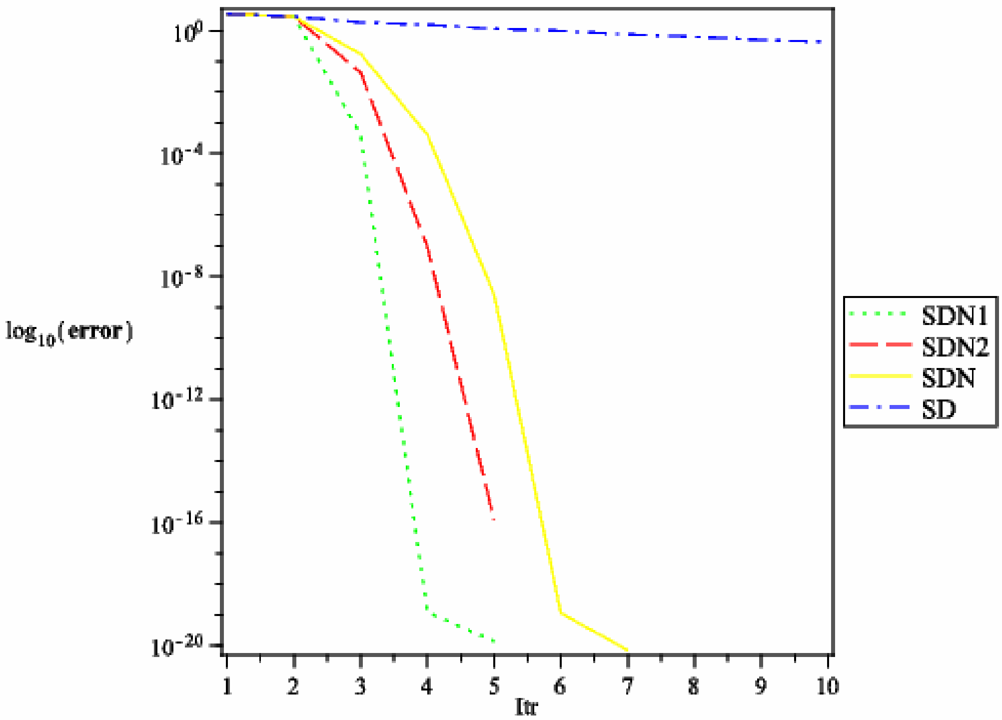

Fig2: The error behaviours of SD and three hybrid methods of Example 2.

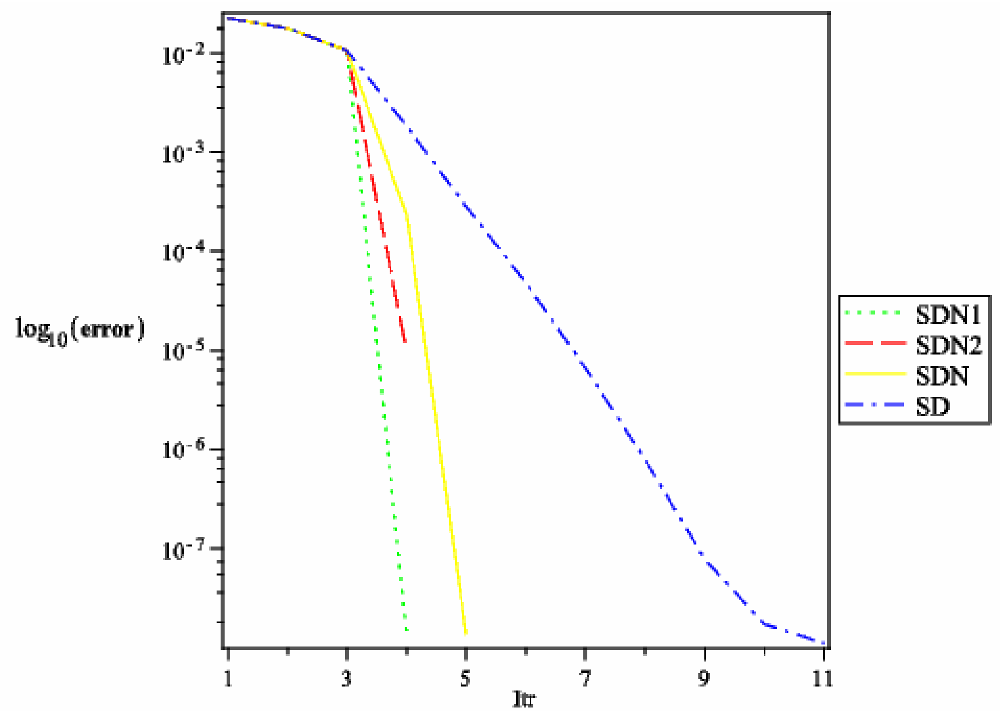

Fig. 3: The errors behaviours of SD and three hybrid methods of Example 3.

3. Conclusion

In this paper, an extension of steepest descent method for solving optimal control problem of Volterra integral equations is presented. The method is restricted to a special class of optimal control problems. Then, with notice of the importance of Newton and two-step Newton methods, SD method is hybridized with these methods. These hybrid methods in comparison with SD are faster and their accuracy is better. The convergence analysis of these methods are also established under some mild assumptions and conditions. One possible future work is to extend this research to optimal control problem with constrained control variables, or use multi-step method with faster rate of convergence.

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