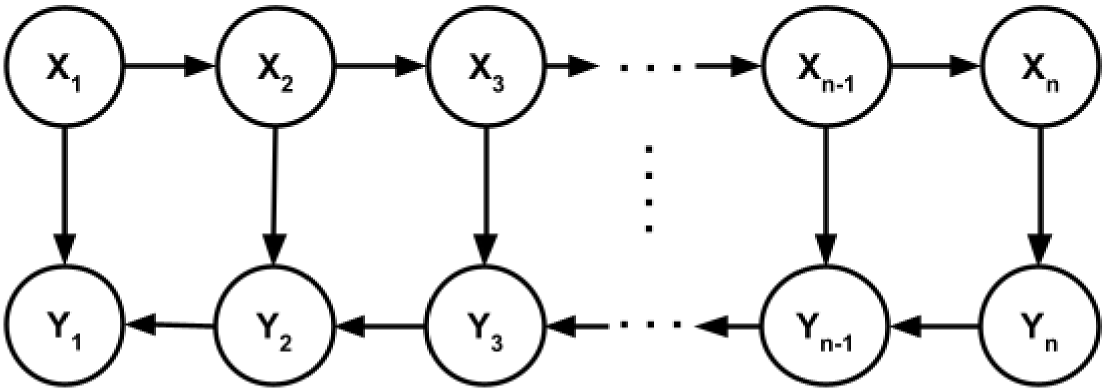
Probabilistic Graphical Models Midterm Exam - Spring 1400 (2021)	Instructor: B. Nasihatkon	دانشگاه صنعتی خواجه نصیرالدین طوسی K. N. TOOSI UNIVERSITY OF TECHNOLOGY

Question 1 - Bayesian Networks (35 points, 15 minutes)

Consider the following Bayesian Network on variables $\mathbf{X_1, X_2, \dots, X_n, Y_1, Y_2, \dots, Y_n}$.



A) Write the joint distribution in terms of the CPDs (7 points).

$\mathbf{P(X_1, X_2, \dots, X_n, Y_1, Y_2, \dots, Y_n) =}$

B) Which of the following statements are **True**, and which are **False** (in general). For each statement, write the word "True" or an Active Trail rejecting the statement. (28 points)

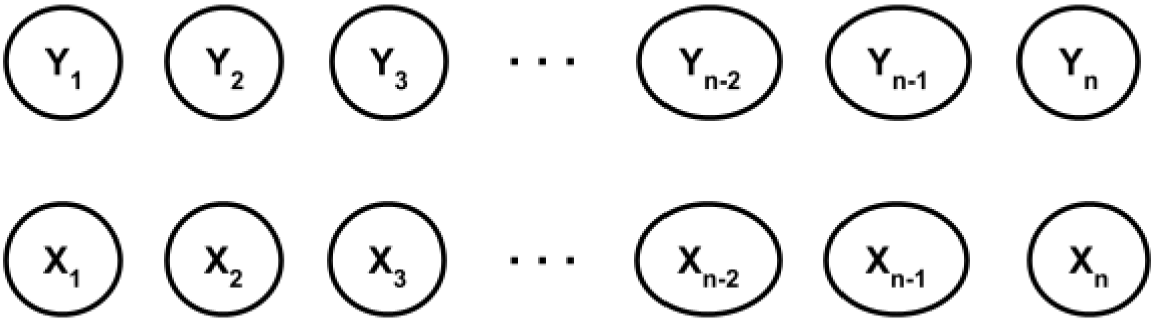
	True/Active Trail		True/Active Trail
$Y_1 \perp Y_3 \mid Y_2$		$Y_n \perp X_{n-1} \mid X_n$	
$X_1 \perp X_3 \mid X_2$		$Y_n \perp X_{n-1} \mid X_n, Y_1$	
$X_1 \perp X_n \mid X_{n-1}, Y_1$		$X_n \perp Y_1 \mid Y_2, Y_{n-1}$	
$X_1 \perp X_n \mid X_{n-1}, Y_{n-1}$		$X_n \perp Y_1 \mid Y_2, X_3$	

Question 2 - MRF (35 points, 30 minutes)

Consider an MRF on variables **$X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n$** with the joint distribution

$$P(X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n) = \frac{1}{Z} \prod_{i=1}^{n-1} exp(1(Y_i = Y_{i+1})) \prod_{i=1}^{n} exp(min(X_i, Y_i)) \prod_{i=1}^{n-1} exp(1(X_i \neq X_{i+1}))$$

a) Draw the edges in the MRF graph below (5 points).



b) Assume that $X_i, Y_i \in \{1, 2, 3, \ldots, m\}$, compute **$Pr(X_1 = Y_1, X_2 = Y_2, \ldots, X_n = Y_n)$**.

(Do not compute the partition function **Z**, just write down the solution in terms of **Z**). (30 pints)

Hint: You cannot write **$Pr(X_1 = Y_1, X_2 = Y_2) = Pr(X_1 = Y_1) Pr(X_2 = Y_2)$** as **$(X_1, Y_1)$** and **$(X_2, Y_2)$** are not independent. What you can do is first compute

$Pr(X_1 = Y_1 = a_1, X_2 = Y_2 = a_2, \ldots, X_n = Y_n = a_n)$

$= P(X_1 = a_1, X_2 = a_2, \ldots, X_n = a_n, Y_1 = a_1, Y_2 = a_2, \ldots, Y_n = a_n)$

Then sum over all combinations of **$a_1, a_2, \ldots, a_n$**, that is $\sum_{a_1=1}^{m} \sum_{a_2=1}^{m} \ldots \sum_{a_n=1}^{m}$

Hint2: $\sum_{i=1}^{m} e^i = e\,(e^m - 1) / (e - 1)$

Continue solution to Question 2.

Question 3 - CRF (30 points, 30 minutes)

Consider a Conditional Random Field defined as

$$P(Y_1, Y_2, \ldots, Y_n \,|\, X_1, X_2, \ldots, X_n) =$$

$$= \frac{1}{Z(X_1, X_2, \ldots, X_n)} \prod_{i=1}^{n-1} exp(1(Y_i = Y_{i+1})) \prod_{i=1}^{n} exp(min(X_i, Y_i)) \prod_{i=1}^{n-1} exp(1(X_i \neq X_{i+1}))$$

on **binary** variables **$X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n \in \{0,1\}$. Show that for $X_1 = X_2 = \ldots = X_n = 0$, the partion function is equal to**

$$Z(X_1, X_2, \ldots, X_n) = Z(0, 0, \ldots, 0) = 2\,(e + 1)^{n-1}$$

Hint: First sum over $\mathbf{Y_1}$, then $\mathbf{Y_2}$, and so on.
Hint 2: Think of the value of $exp(1(Y = 0)) + exp(1(Y = 1))$ for Y=0 and Y=1.