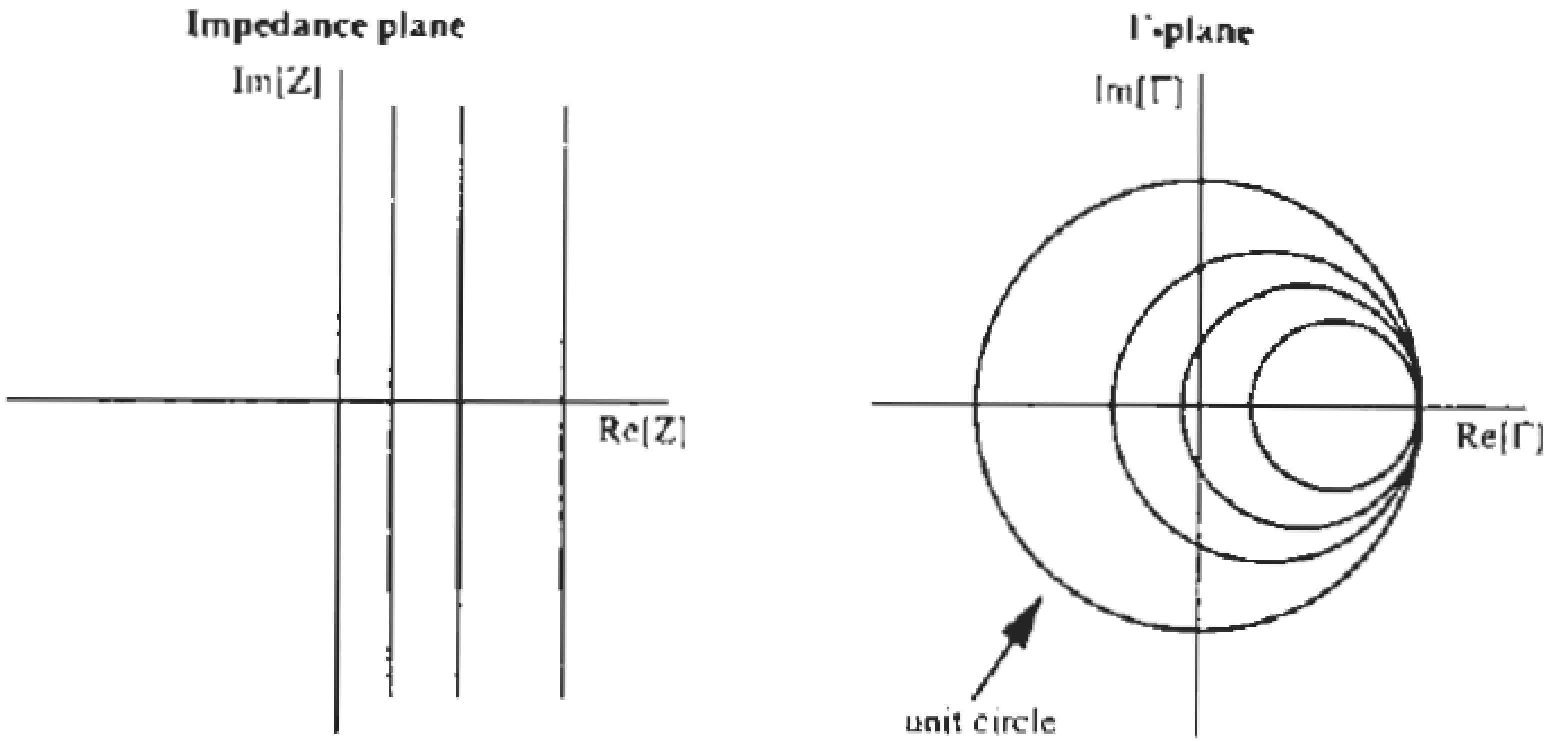
The Smith Chart and S-Parameters

Reflection Coefficient

$$\Gamma = \frac{z_l - z_o}{z_l + z_o} = \frac{\dfrac{z_l}{z_o} - 1}{\dfrac{z_l}{z_o} + 1} = \frac{z_{nl} - 1}{z_{nl} + 1}$$

Constant Resistance



Mapping of constant-resistance lines in Z-plane to circles in Γ-plane.

Proof:

assume: $Z_0 = R_0 + JX_0$, $Z_L = R_L + JX_L \Rightarrow z_{nl} = r_l + jx_l = Z_L / Z_0$

$$\Gamma = \frac{z_l - z_o}{z_l + z_o} = \frac{\dfrac{z_l}{z_o} - 1}{\dfrac{z_l}{z_o} + 1} = \frac{z_{nl} - 1}{z_{nl} + 1}$$

$$z_{nl}\Gamma + \Gamma = z_{nl} - 1$$

$$z_{nl} = \frac{1+\Gamma}{1-\Gamma}$$

$$\Rightarrow \quad r_l + jx_l = \frac{1+\Gamma_r + j\Gamma_i}{1-\Gamma_r - j\Gamma_i}$$

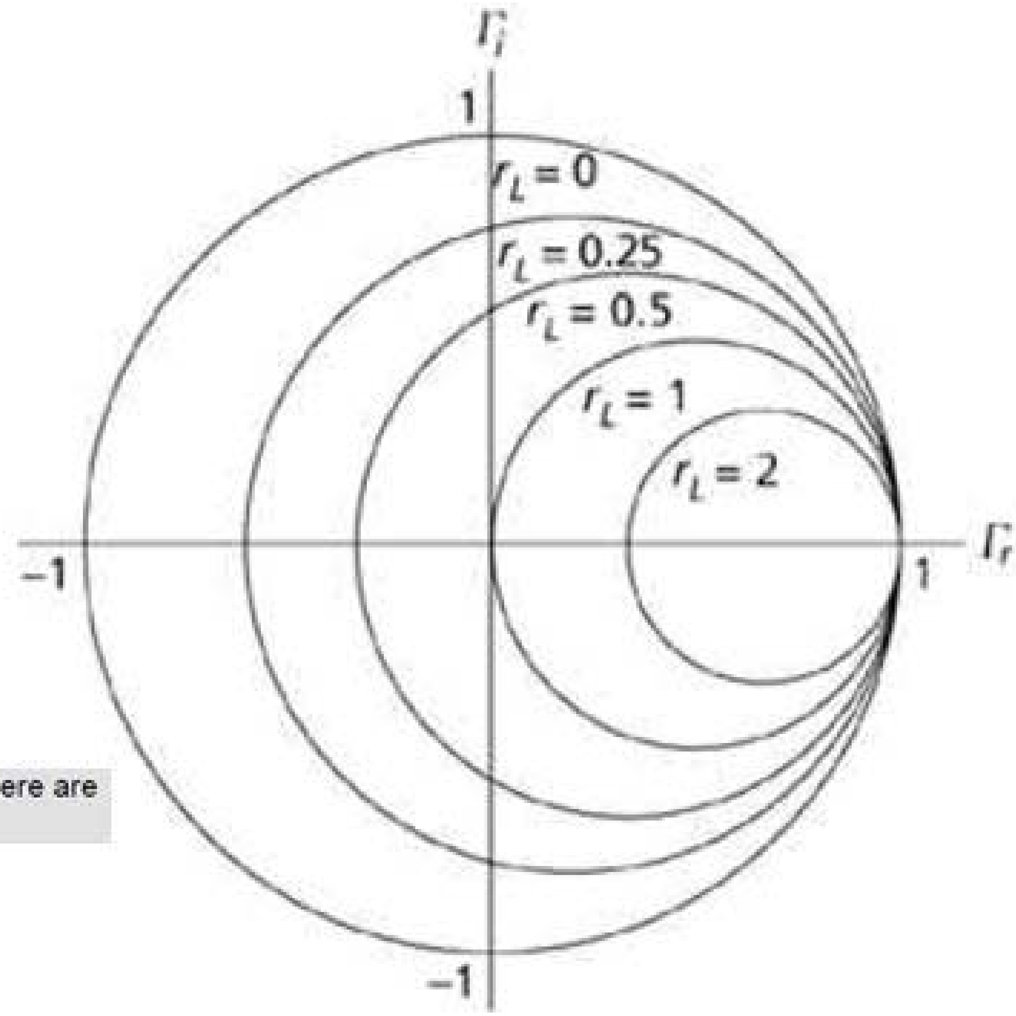
$$r_l + jx_l = \frac{1+\Gamma_r + j\Gamma_i}{1-\Gamma_r - j\Gamma_i}$$

$$r_l + jx_l = \frac{1-\Gamma_r^{\;2} - \Gamma_i^{\;2}}{(1-\Gamma_r)^2 + \Gamma_i^{\;2}} + j\frac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^{\;2}}$$

$$r_l = \frac{1-\Gamma_r^{\;2} - \Gamma_i^{\;2}}{\Gamma_r^{\;2} - 2\Gamma_r + 1 + \Gamma_i^{\;2}}$$

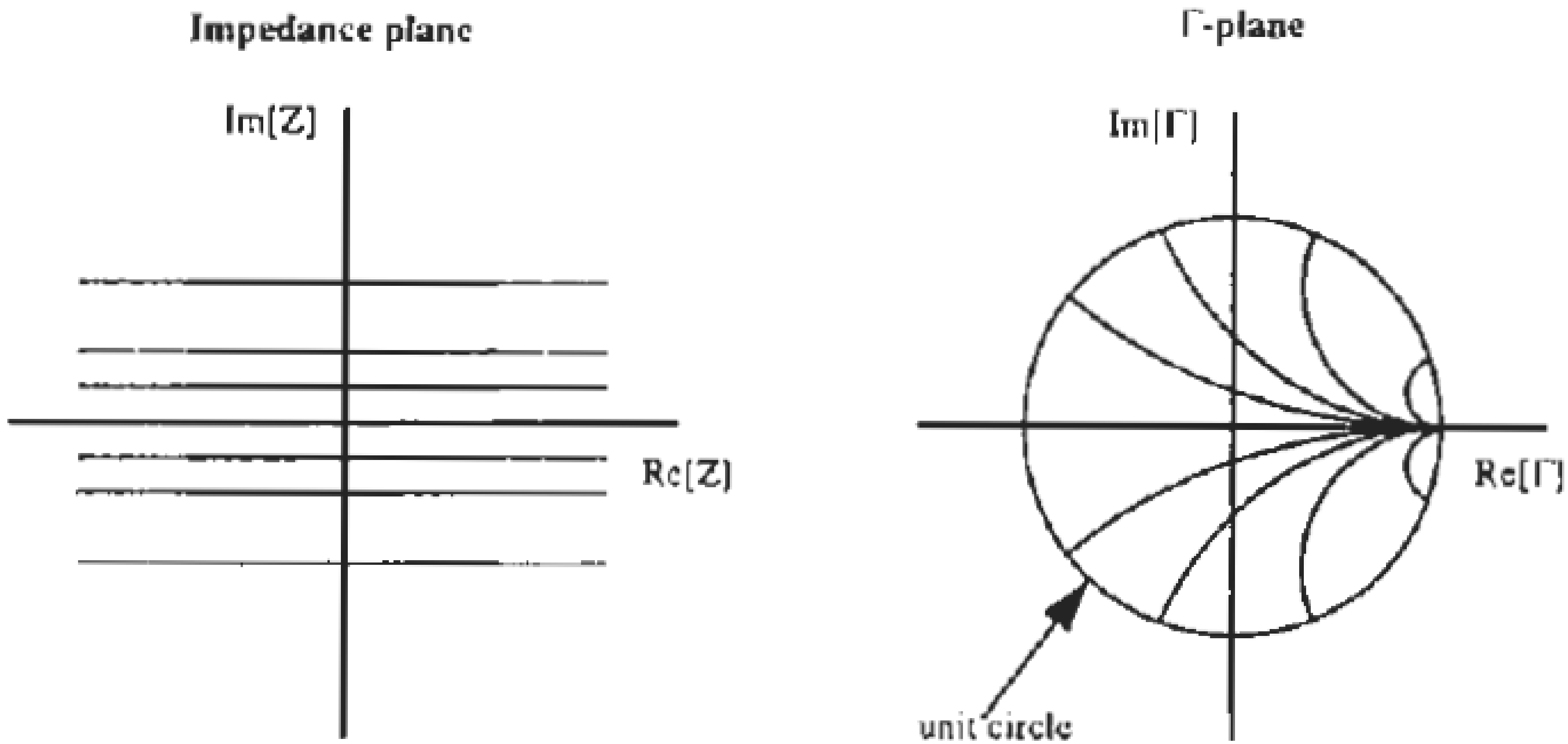
$$(x-a)^2 + (y-b)^2 = r^2$$

$$\left[\Gamma_r - \frac{r_l}{r_l+1}\right]^2 + \left[\Gamma_i - 0\right]^2 = \left[\frac{1}{r_l+1}\right]^2$$



Points of constant resistance form circles on the complex reflection-coefficient plane. Shown here are the circles for various values of load resistance.

Constant Reactance



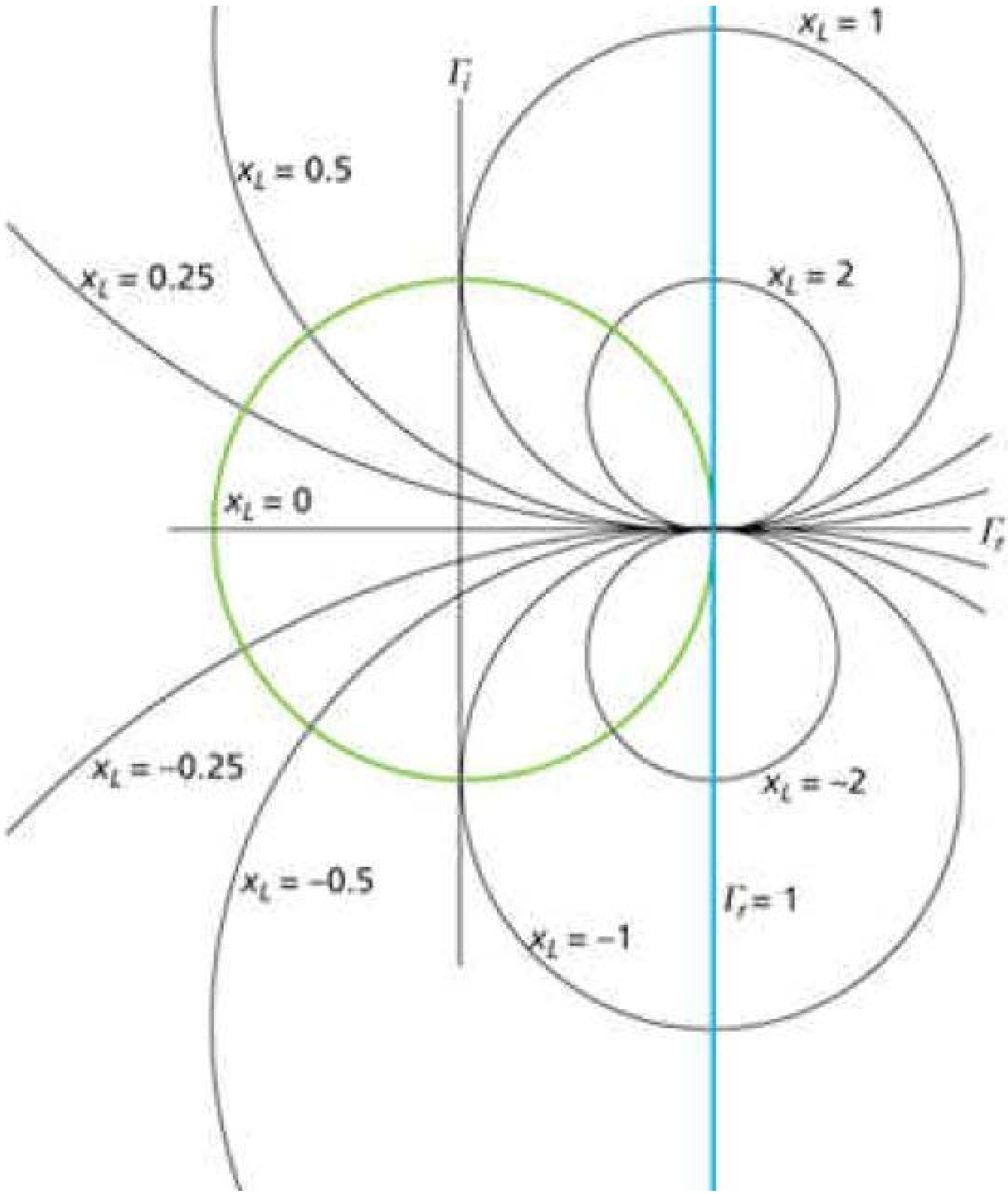
Mapping of constant-reactance lines in Z-plane to contours in Γ-plane.

$$r_l + jx_l = \frac{1-\Gamma_r^{\,2}-\Gamma_i^{\,2}}{(1-\Gamma_r)^2+\Gamma_i^{\,2}} + j\frac{2\Gamma_i}{(1-\Gamma_r)^2+\Gamma_i^{\,2}}$$

$$x_l = \frac{2\Gamma_i}{1-2\Gamma_r+\Gamma_r^{\,2}+\Gamma_i^{\,2}}$$

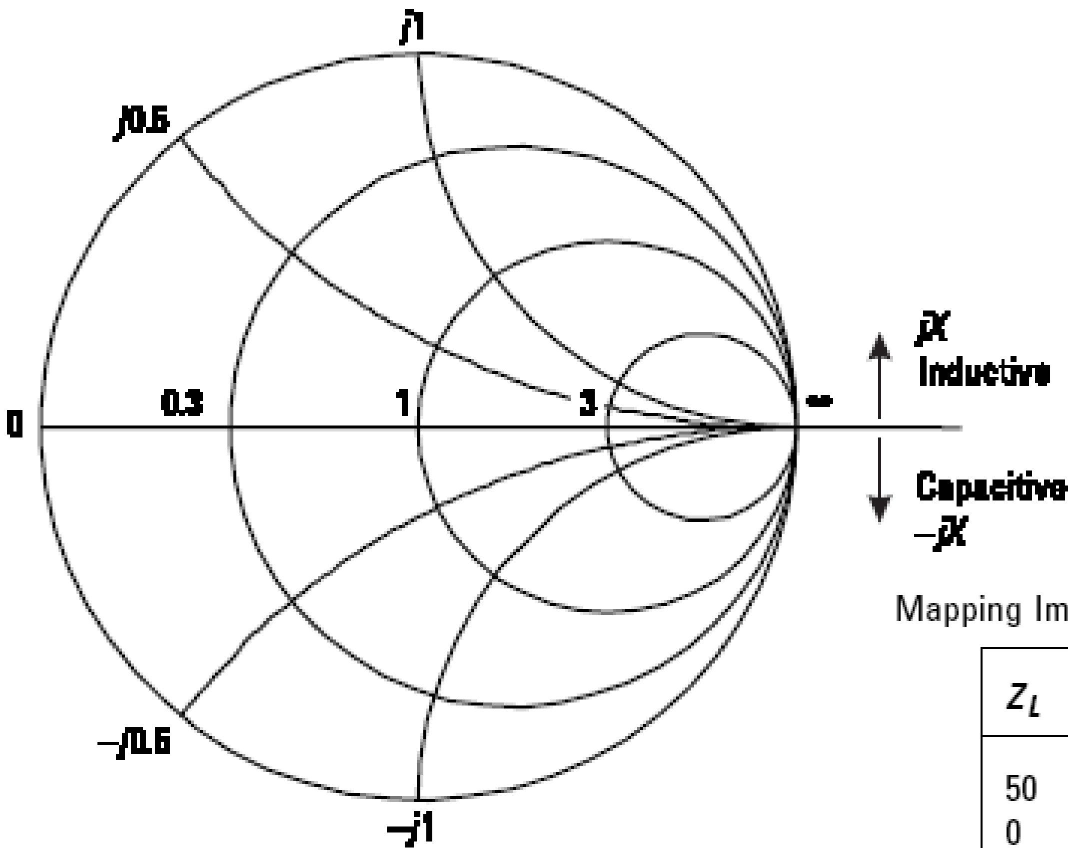
$$(x-a)^2+(y-b)^2=r^2$$

$$(\Gamma_r-1)^2+(\Gamma_i-\frac{1}{x_l})^2=(\frac{1}{x_l})^2$$



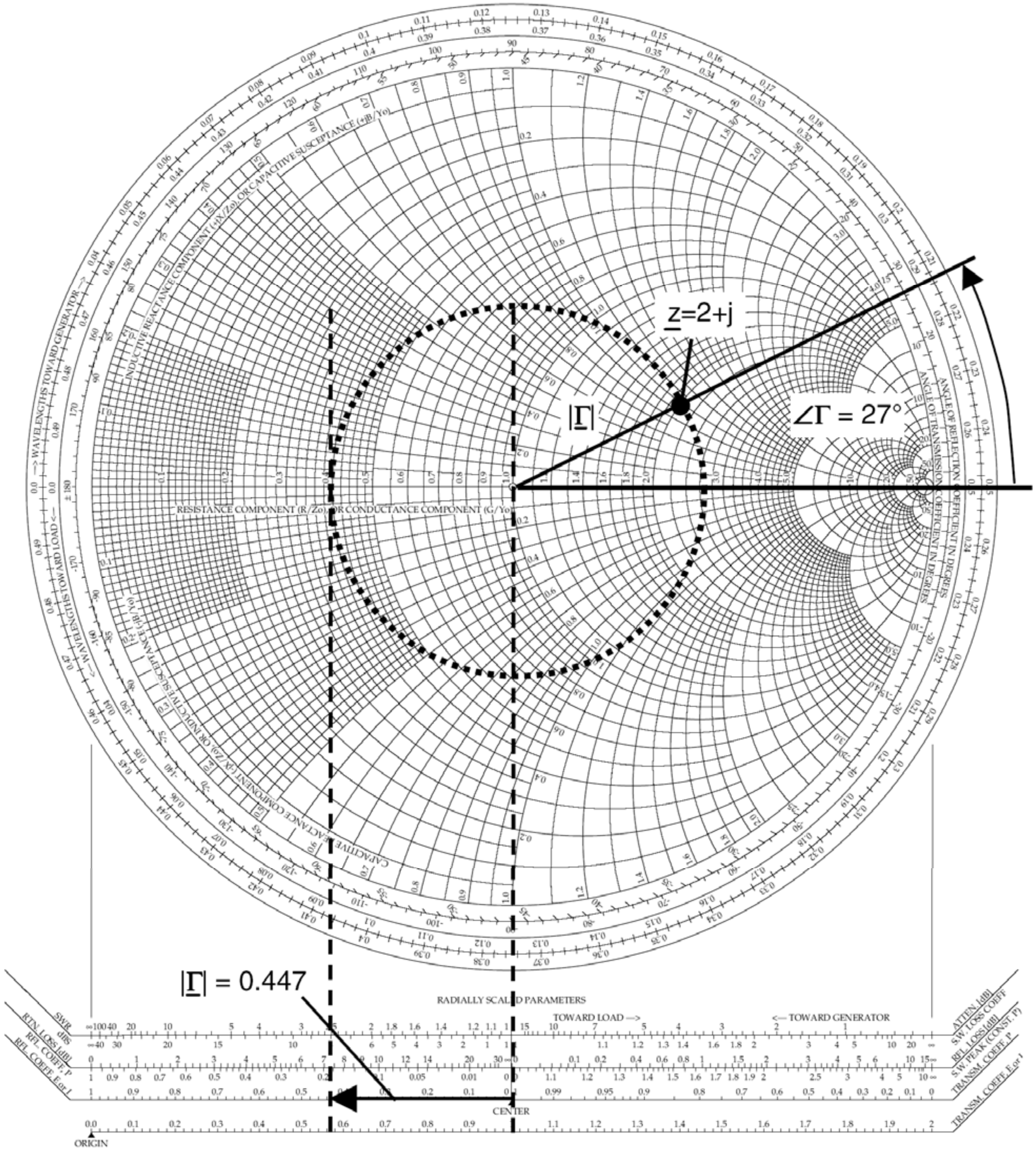
$x_L = 1$
Γ_i
$x_L = 0.5$
$x_L = 0.25$
$x_L = 2$
$x_L = 0$
Γ_r
$x_L = -0.25$
$x_L = -2$
$x_L = -0.5$
$\Gamma_r = 1$
$x_L = -1$

A Z Smith Chart



Mapping Impedances to Points on the Smith Chart

Z_L	Γ
50	0
0	−1
∞	1
100	0.333
25	−0.333
$j50$	$1\angle 90°$
jX	$1\angle 2\tan^{-1}(X/50)$
$50 - j141.46$	$0.8166\angle -35.26°$



z=2+j
|Γ|
∠Γ = 27°
|Γ| = 0.447
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
RADIALLY SCALED PARAMETERS
TOWARD LOAD —>
<— TOWARD GENERATOR
CENTER
ORIGIN

For this simple example we can calculate

$$\underline{\Gamma} = \frac{\underline{Z} - Z_0}{\underline{Z} + Z_0} = \frac{\underline{z} - 1}{\underline{z} + 1} = \frac{2 + j - 1}{2 + j + 1} = \frac{1 + j}{3 + j}\,.$$

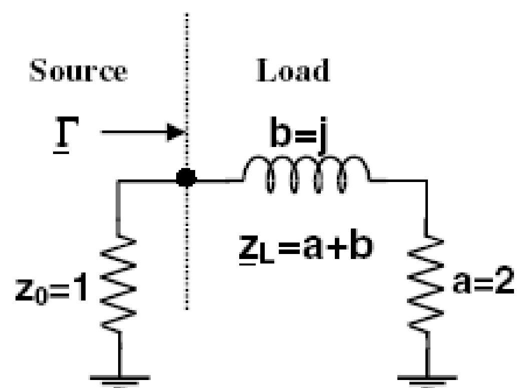
Separated into magnitude and phase, $\underline{\Gamma}$ is given by

$$\left|\underline{\Gamma}\right| = 0.447$$

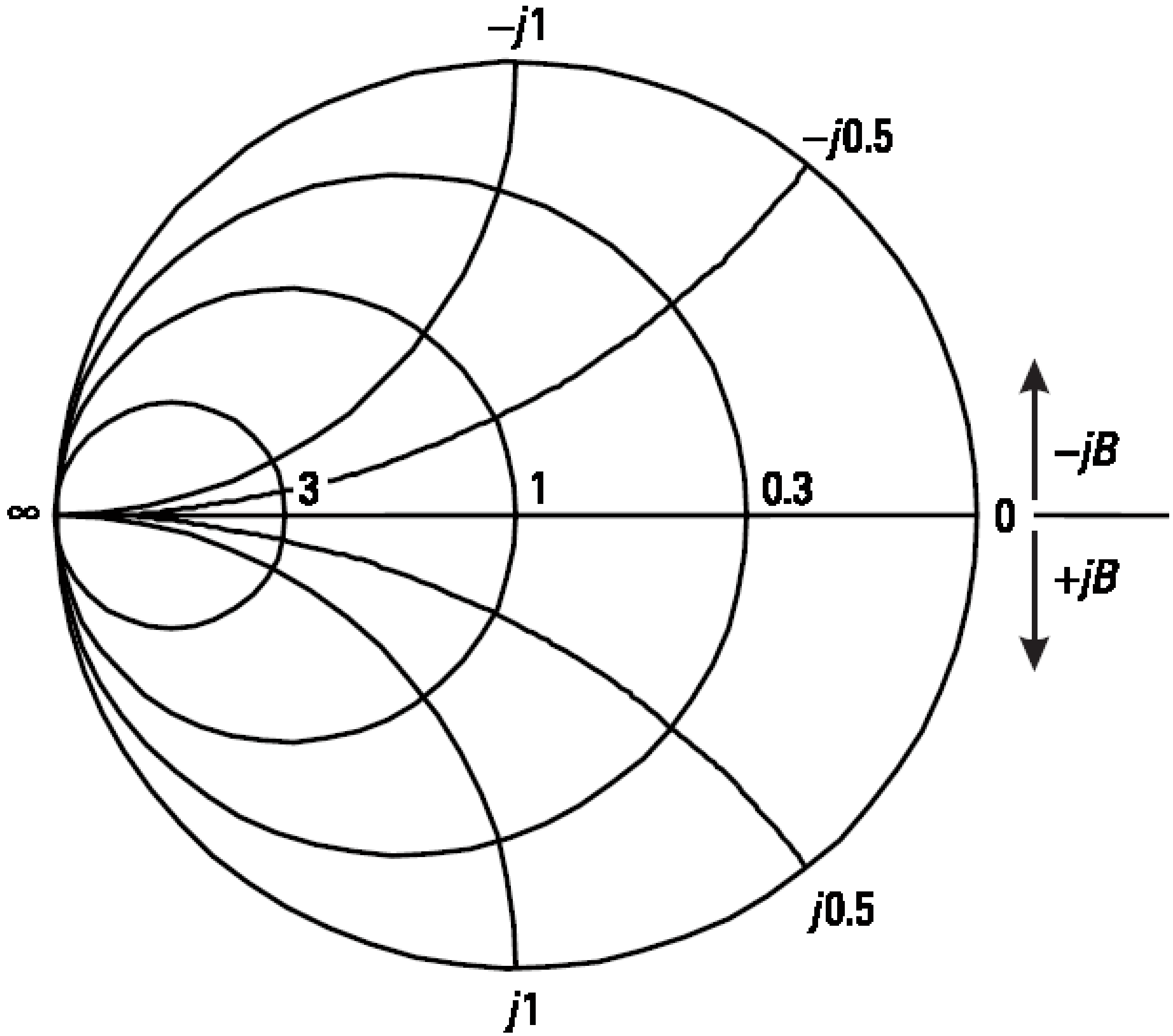
and

$$\angle\Gamma = \arctan(1) - \arctan\left(1/3\right) = 27^\circ\,.$$

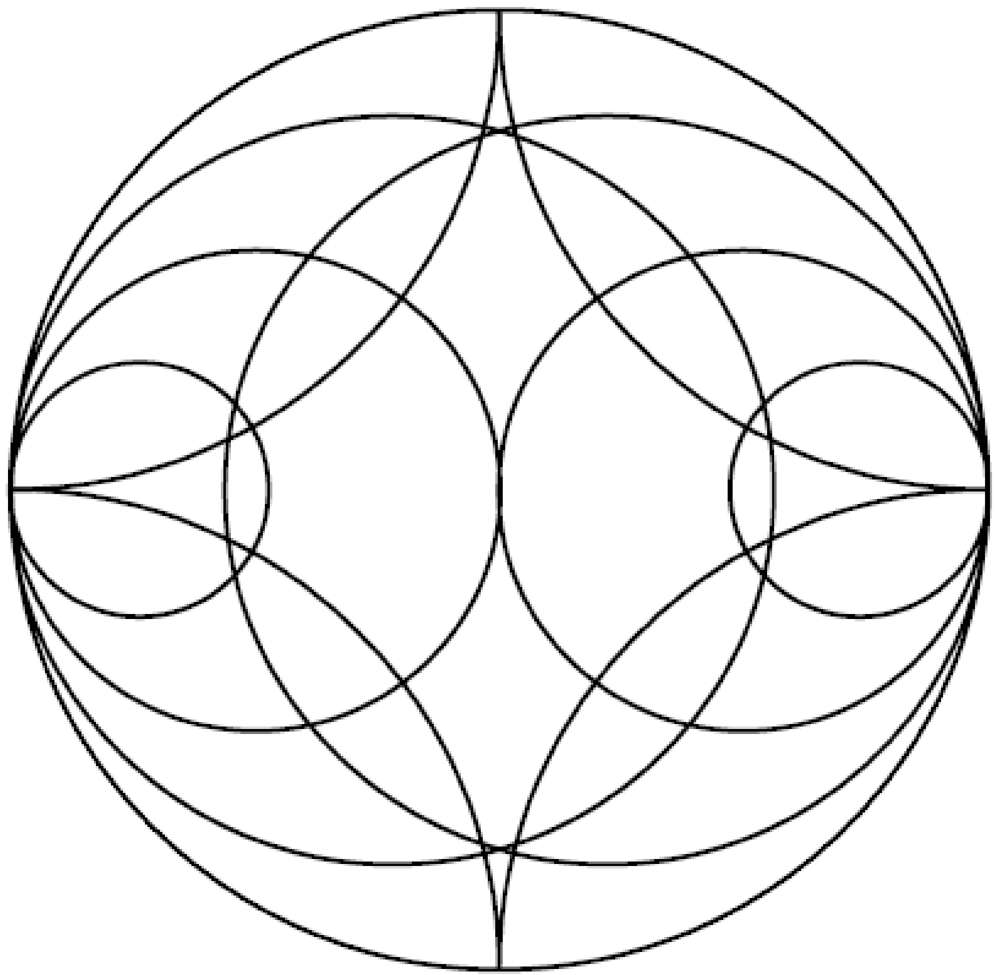
Consequently, the magnitude of the reflected power is proportional to $\left|\underline{\Gamma}^2\right|$. In our example, 20% of the power is reflected at the impedance transition, whereas the remaining 80% ($1 - \left|\underline{\Gamma}^2\right|$) is fed into the load.

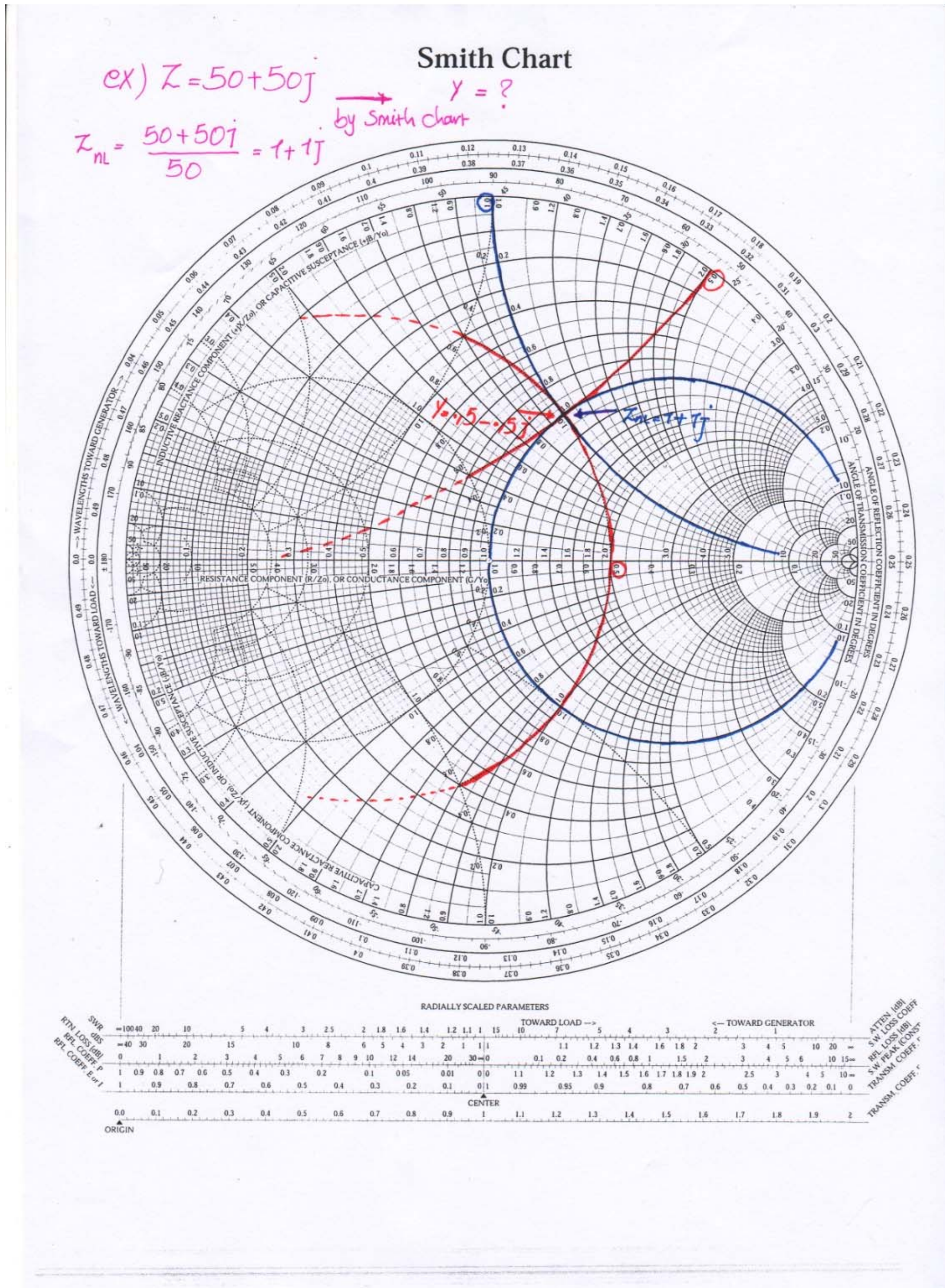


An Admittance or Y Smith Chart



A ZY Smith Chart

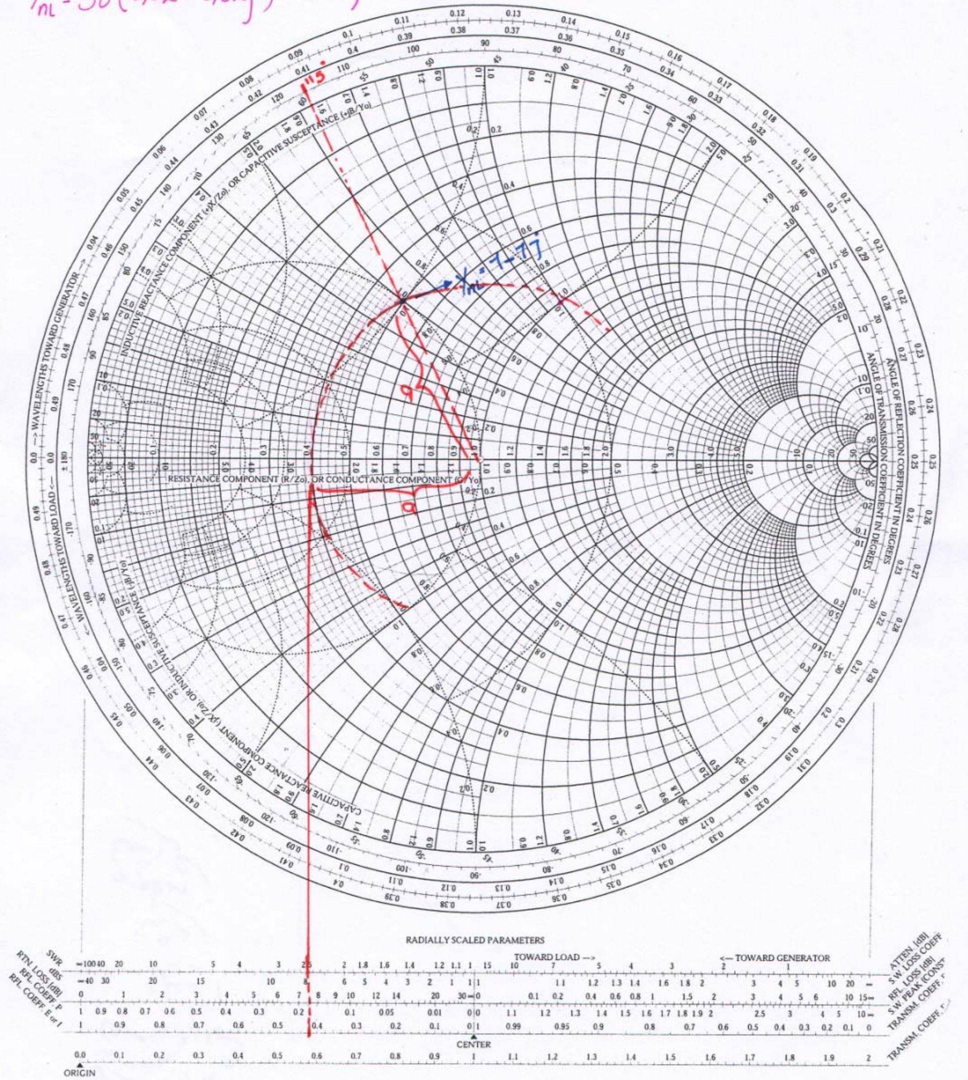



Smith Chart
ex) Z = 50+50j → Y = ?
by Smith chart
Znl = (50+50j)/50 = 1+1j
Y=.5-.5j
Znl=1+1j
RADIALLY SCALED PARAMETERS
TOWARD LOAD →
← TOWARD GENERATOR
CENTER
ORIGIN

Smith Chart

ex) $Y = 0{,}02 - 0{,}02j \rightarrow \Gamma = ?$

$Y_{nL} = 50(0{,}02 - 0{,}02j) = 1 - 1j \Rightarrow \Gamma = 0{,}47 \angle 115°$

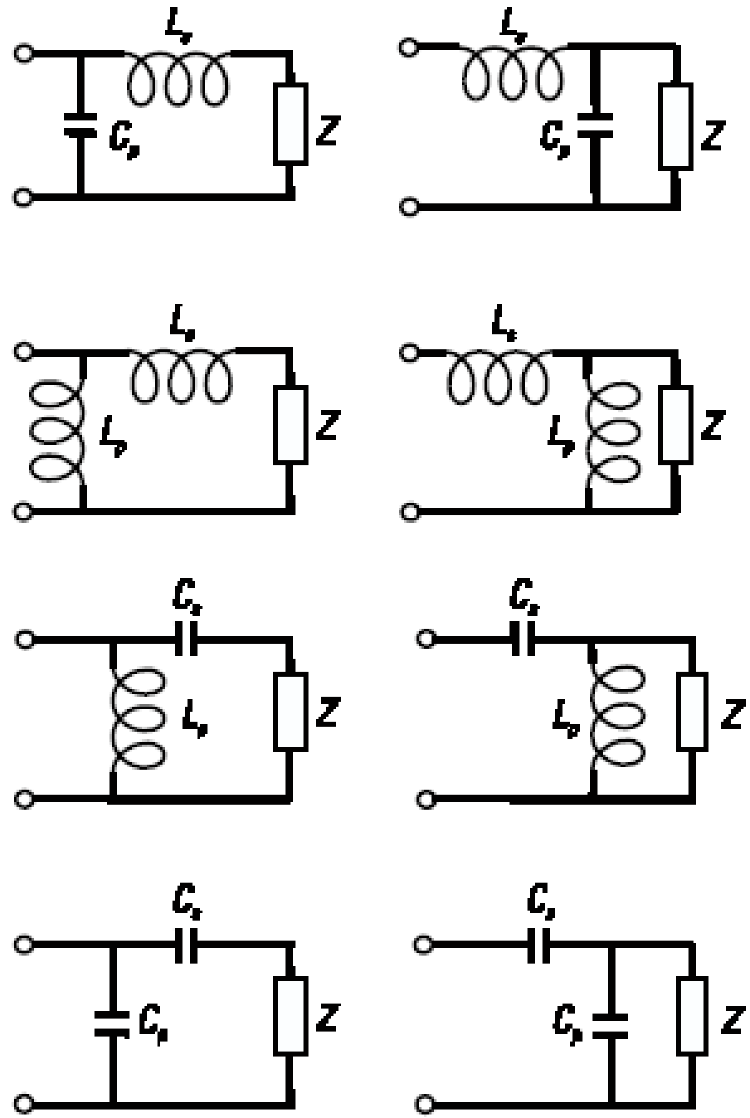


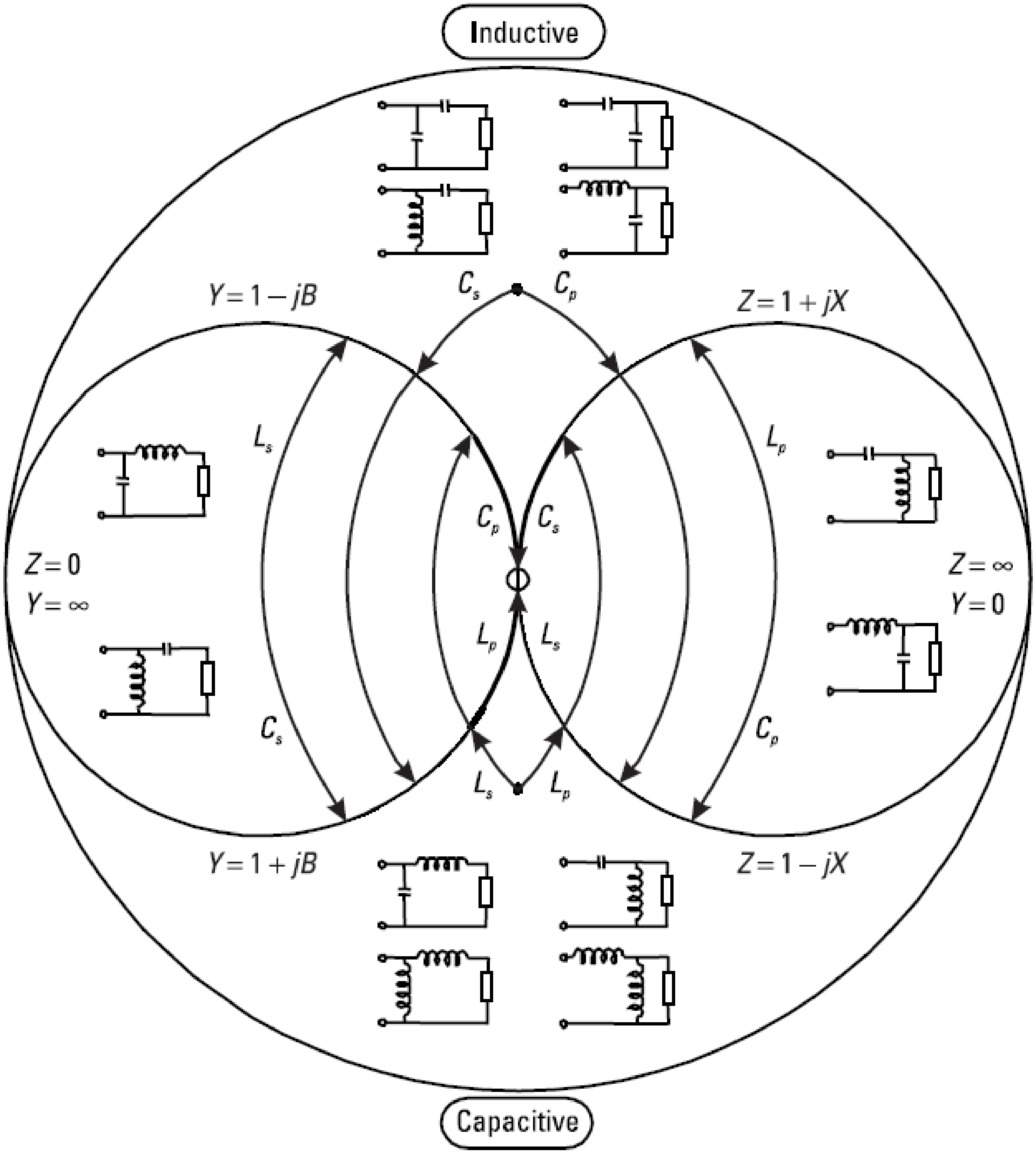
Impedance Matching

Using Lumped Components to Match Circuits

Component Added	Effect	Description of Effect
Series inductor	$z \rightarrow z + j\omega L$	Move clockwise along a resistance circle
Series capacitor	$z \rightarrow z - j/\omega C$	Smaller capacitance increases impedance ($-j/\omega C$) to move counterclockwise along a resistance circle
Parallel inductor	$y \rightarrow y - j/\omega L$	Smaller inductance increases admittance ($-j/\omega L$) to move counterclockwise along a conductance circle
Parallel capacitor	$y \rightarrow y + j\omega C$	Move clockwise along a conductance circle

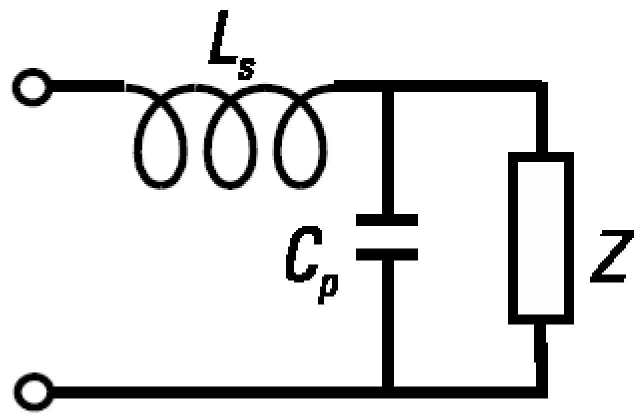
Eight Possible Impedance-Matching Networks




Inductive
C_s
C_p
$Y = 1 - jB$
$Z = 1 + jX$
L_s
L_p
C_p
C_s
$Z = 0$
$Y = \infty$
$Z = \infty$
$Y = 0$
L_p
L_s
C_s
C_p
L_s
L_p
$Y = 1 + jB$
$Z = 1 - jX$
Capacitive

Example#1

Match $Z = 150 - 50j$ to 50Ω using the techniques just developed.



Normalized Impedance: $3 - 1j$

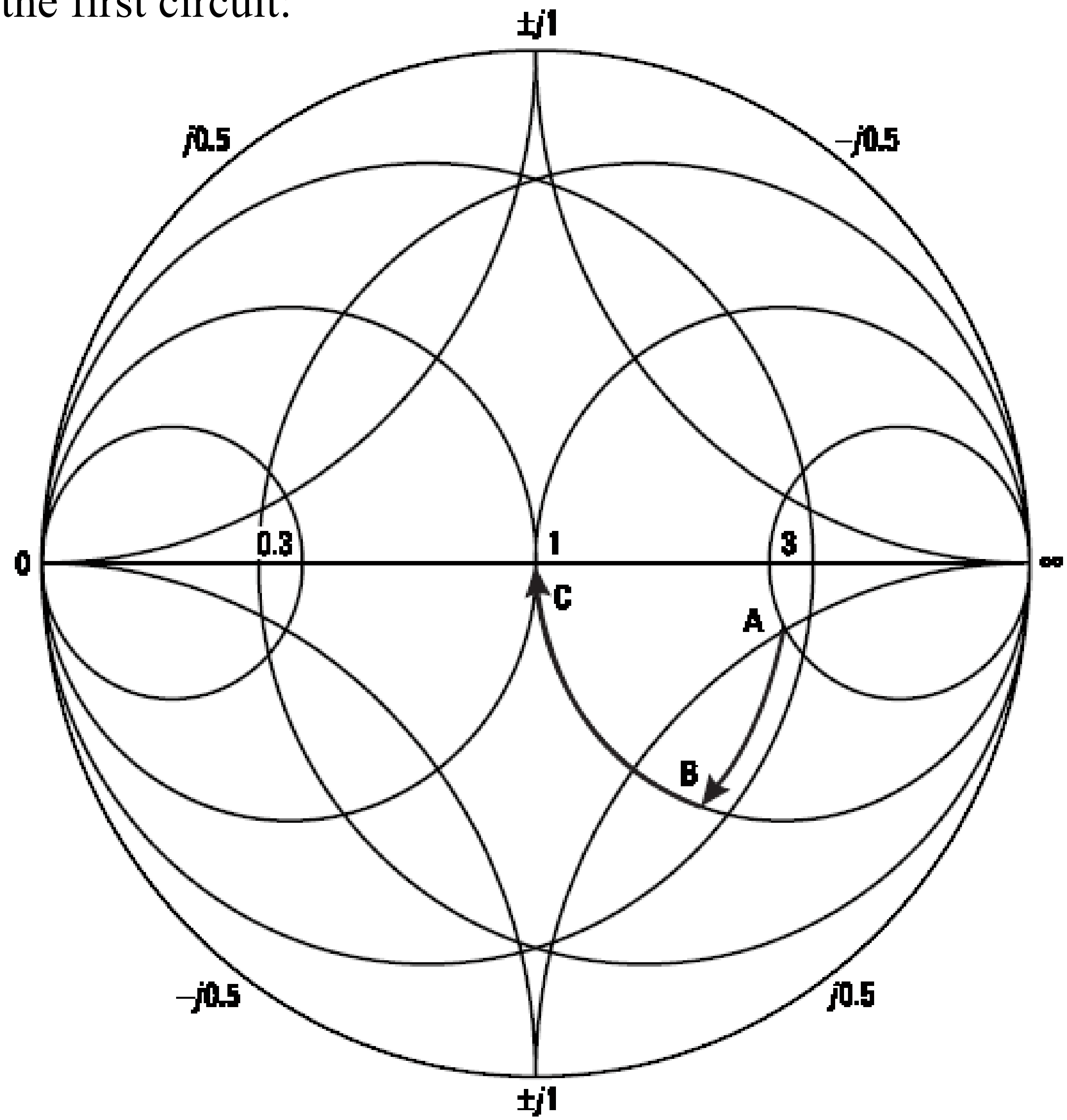
$Y_A = 1/Z_A = 0.3 + j0.1$

$Y_B = 0.3 + 0.458j$ we need a capacitor admittance of $0.348j$.

Since $Z_B = 1/Y_B = 1 - 1.528j$, an inductor reactance of $1.528j$ is needed to bring it to the center.

Cont’d

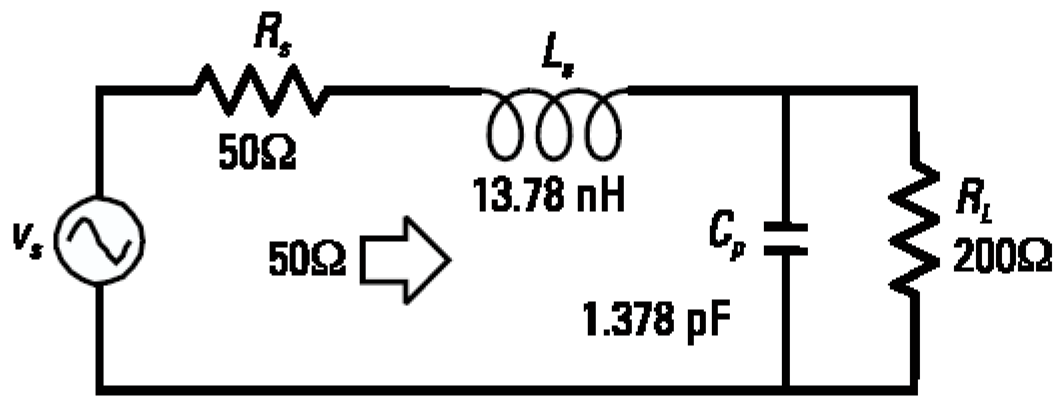
Match process of the first circuit:



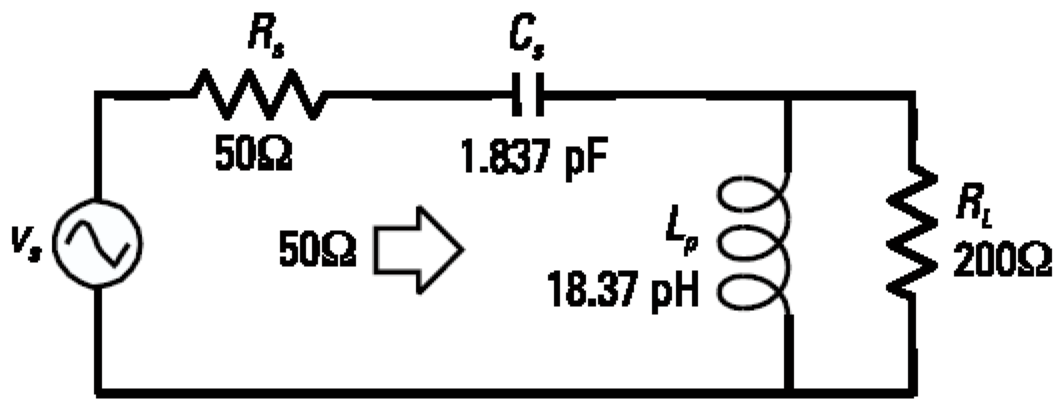
Example#2

Match a 200-Ω load to a 50-Ω source at 1 GHz

Low-pass structure

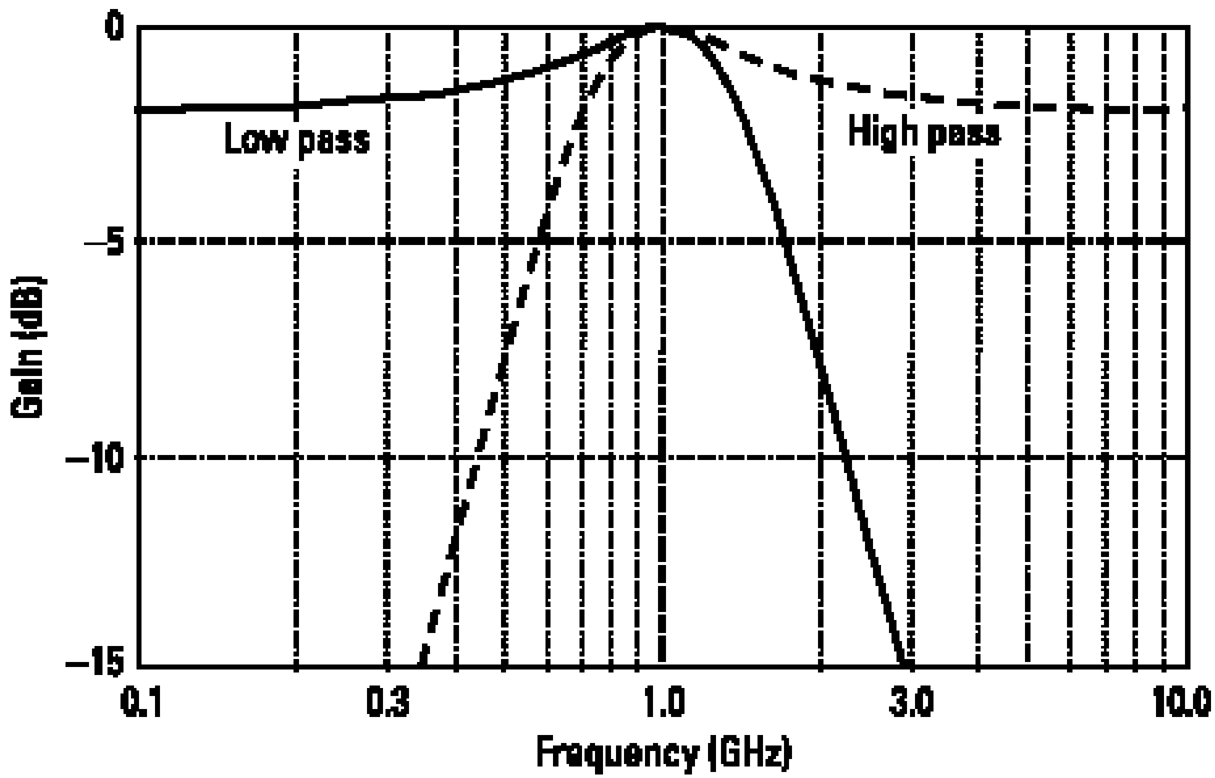


High-pass structure



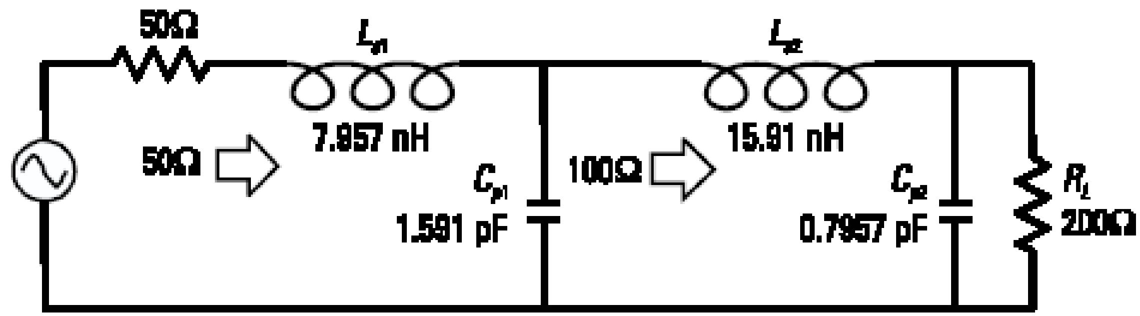
Cont'd

Frequency response for lowpass and highpass matching networks.

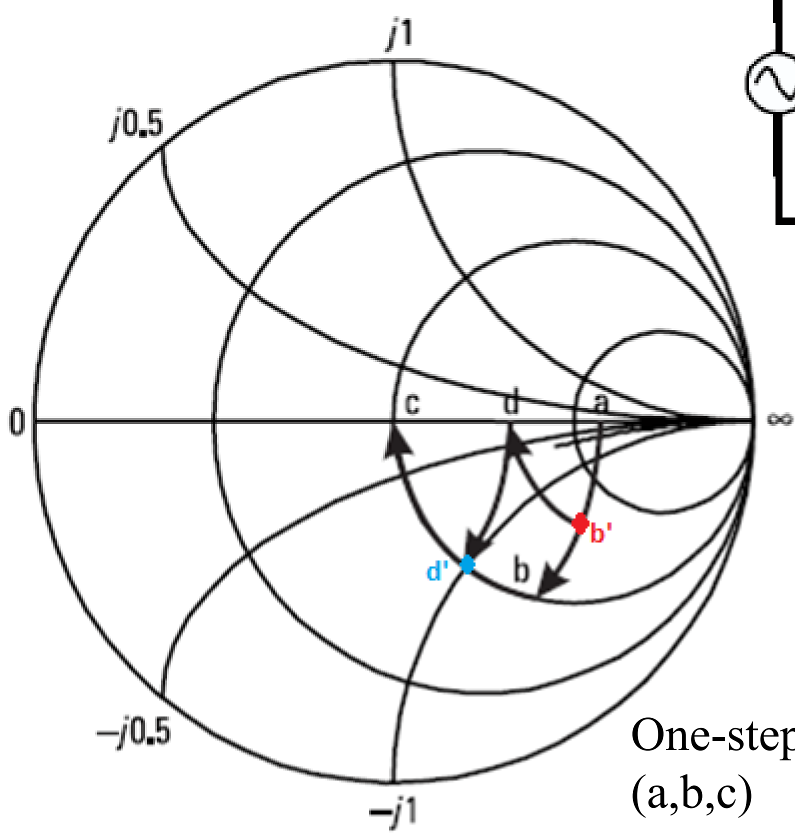


Example#3

Match 200Ω to 50Ω at 1 GHz using an ell matching network. Do it first in one step, then do it in two steps matching it first to 100Ω. Compare the bandwidth of the two matching networks.

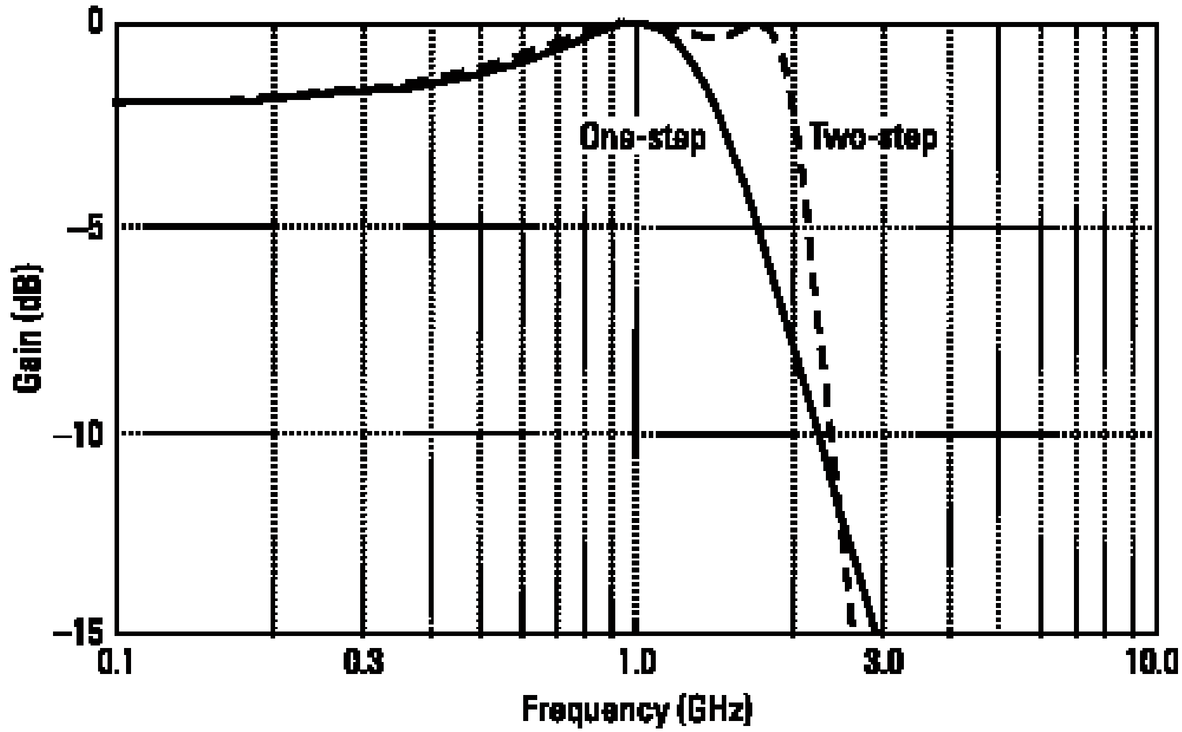


Two-step matching (a,b',d,d',c)

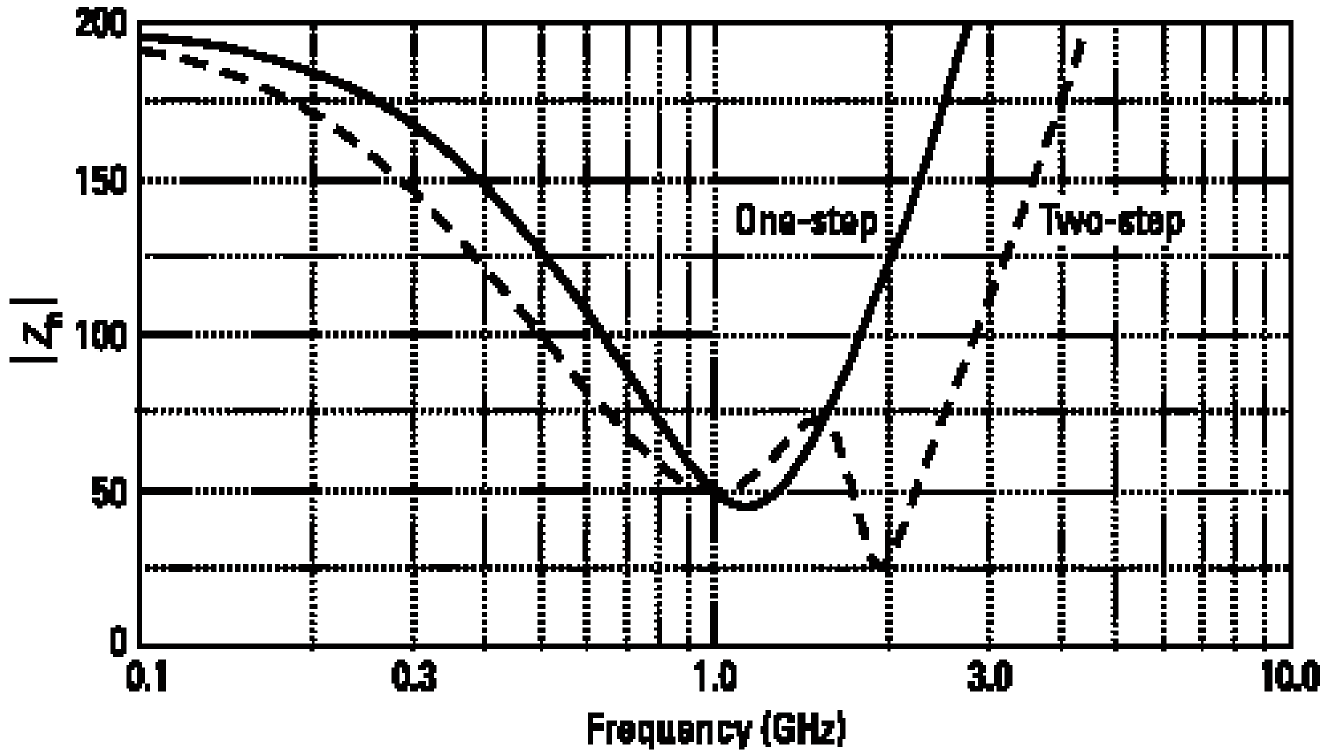


One-step matching is performed in the previous example. (a,b,c)

Cont’d

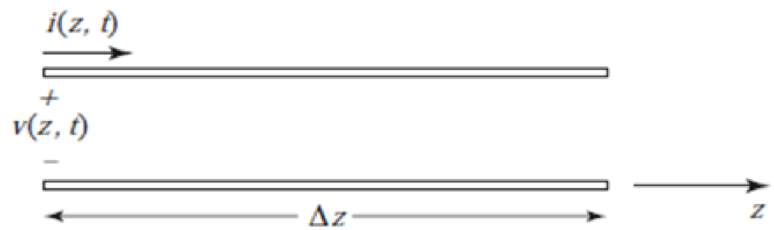


Cont’d



A comparison of frequency response shown in above figures clearly shows the bandwidth broadening effect of matching in two steps.

Circuit Model for Transmission Line



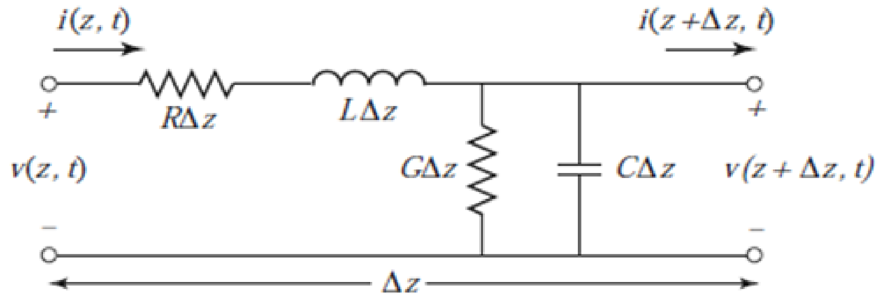
(a)

R: (Ω/m)

L: (H/m)

C: (F/m)

G: (S/m)



(b)

KVL: $v(z,t) - R\Delta z\, i(z,t) - L\Delta z \frac{\partial i(z,t)}{\partial t} - v(z+\Delta z,t) = 0$ ⇒ $\frac{\partial v(z,t)}{\partial z} = -Ri(z,t) - L\frac{\partial i(z,t)}{\partial t}$

KCL: $i(z,t) - G\Delta z\, v(z+\Delta z,t) - C\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t} - i(z+\Delta z,t) = 0$ ⇒ $\frac{\partial i(z,t)}{\partial z} = -Gv(z,t) - C\frac{\partial v(z,t)}{\partial t}$

Sinusoidal steady-State Analysis:

$$\frac{dV(z)}{dz} = -(R+j\omega L)I(z)$$

$$\frac{dI(z)}{dz} = -(G+j\omega C)V(z)$$

Cont'd

- $\frac{dV(z)}{dz} = -(R+j\omega L)I(z)$
  $\frac{dI(z)}{dz} = -(G+j\omega C)V(z)$

$\Rightarrow$

$$\frac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0$$

$$\frac{d^2I(z)}{dz^2} - \gamma^2 I(z) = 0$$

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$V(z) = V_0^{+} e^{-\gamma Z} + V_0^{-} e^{\gamma Z}$$
$$I(z) = I_0^{+} e^{-\gamma Z} + I_0^{-} e^{\gamma Z}$$

or

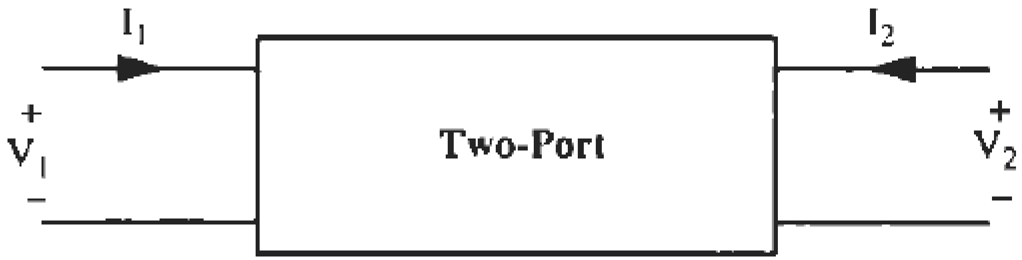
$$V(z) = V_i e^{-\gamma Z} + V_r e^{\gamma Z}$$
$$I(z) = I_i e^{-\gamma Z} + I_r e^{\gamma Z}$$

It is easily proven that: $I(z) = \frac{\gamma}{R+j\omega L}(V_0^{+} e^{-\gamma Z} - V_0^{-} e^{\gamma Z})$

Defining: $Z_0 = \frac{R+j\omega L}{\gamma} = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$ $\Rightarrow$ $I(z) = \frac{V_0^{+}}{Z_0} e^{-\gamma Z} - \frac{V_0^{-}}{Z_0} e^{\gamma Z}$

$$v(z,t) = |V_0^{+}|\cos(\omega t - \beta z + \varphi^{+})e^{-\alpha z} + |V_0^{-}|\cos(\omega t + \beta z + \varphi^{-})e^{\alpha z}$$

Z-Parameters

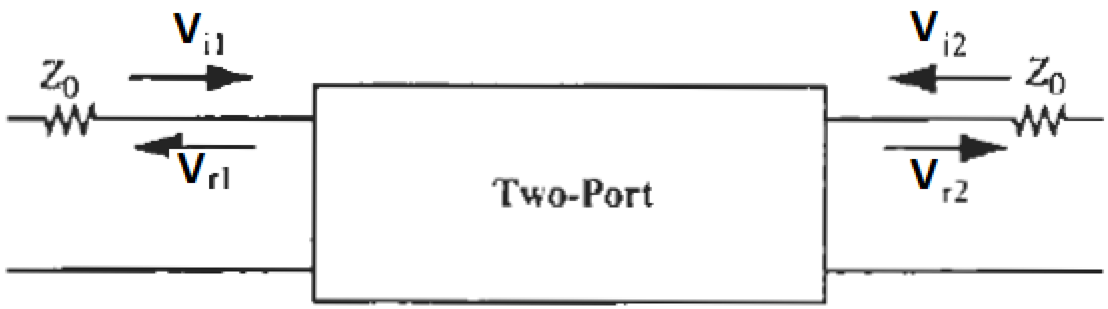


Port variable definitions.

$$V_1 = Z_{11} I_1 + Z_{12} I_2,$$

$$V_2 = Z_{21} I_1 + Z_{22} I_2.$$

S-Parameters



S-parameter port variable definitions.

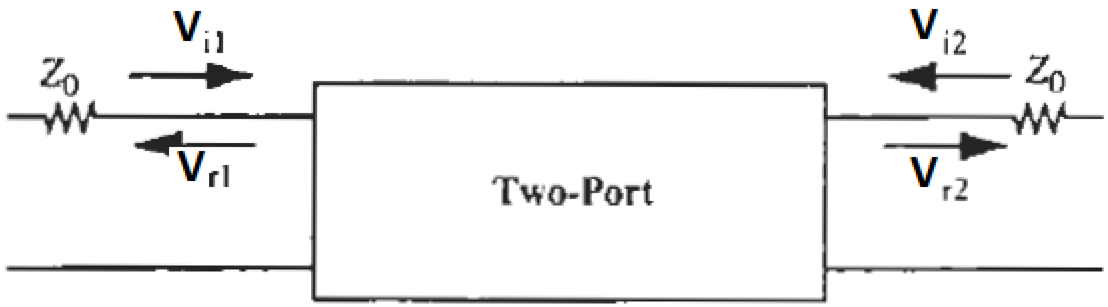
$$V(z) = V_i e^{-\gamma z} + V_r e^{\gamma z}$$

$$a_1 = \frac{V_{i1}}{\sqrt{Z_0}}, a_2 = \frac{V_{i2}}{\sqrt{Z_0}}, b_1 = \frac{V_{r1}}{\sqrt{Z_0}}, b_2 = \frac{V_{r2}}{\sqrt{Z_0}}$$

$$b_1 = s_{11}a_1 + s_{12}a_2$$
$$b_2 = s_{21}a_1 + s_{22}a_2$$

The normalization by the square root of Z_0 is a convenience that makes the square of the magnitude of the various a_n and b_n equal to the power of the corresponding incident or reflected wave.

S-Parameters



S-parameter port variable definitions.

Driving the input port with the output port terminated in Z_0 sets a_2 equal to zero, and allows us to determine the following parameters:

$$s_{11} = \frac{b_1}{a_1} = \frac{\mathbf{V}_{r1}}{\mathbf{V}_{i1}} = \Gamma_1,$$

$$s_{21} = \frac{b_2}{a_1} = \frac{\mathbf{V}_{r2}}{\mathbf{V}_{i1}}.$$

Thus, s_{11} is simply the input reflection coefficient, while s_{21} is a sort of gain since it relates an output wave to an input wave. Specifically, its magnitude squared is called the forward transducer power gain with Z_0 as source and load impedance.