IET Generation, Transmission & Distribution

Research Article

ISSN 1751-8687
Received on 25th March 2015
Revised on 17th October 2015
Accepted on 16th November 2015
doi: 10.1049/iet-gtd.2015.0388
www.ietdl.org

Linear approximated formulation of AC optimal power flow using binary discretisation

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Abstract: This study presents a novel linear approximated methodology for full alternating current-optimal power flow (AC-OPF). The AC-OPF can provide more precise and real picture of full active and reactive power flow modelling, along with the voltage profile of buses compared to the commonly used direct current-optimal power flow. While the AC-OPF is a non-linear programming problem, this can be transformed into a mixed-integer linear programming environment by the proposed model without loss of accuracy. The global optimality of the solution for the approximated model can be guaranteed by existing algorithms and software. The numerical results and simulations which represent the effectiveness and applicability of the proposed model are given and completely discussed in this study.

Nomenclature

Indices

i, j	indices of buses
k	indices for binary variables

Sets

Ωb	set of all buses

Constants

a_i, b_i, c_i	cost coefficients of active power generation at bus i (\$/MW^{2}h, \$/MWh, \$/h)
$\mathrm{B}M1_{ij}$, $\mathrm{B}M2_{ij}$	disjunctive parameters (big M parameters)
jB_{ij}^{sh}	shunt admittance of line ij (℧)
k_1	maximum value of k
n	number of sides of regular polygon used in piecewise linear formulation of a circle
PD_i	active power demand at bus i (MW)
$\mathrm{PG}_i^{\min}$	minimum active power generation at bus i (MW)
$\mathrm{PG}_i^{\max}$	maximum active power generation at bus i (MW)
QD_i	reactive power demand at bus i (MVAr)
$QG_i^{\min}$	minimum reactive power generation at bus i (MVAr)
$QG_i^{\max}$	maximum reactive power generation at bus i (MVAr)
$\lvert S_{ij}\rvert^{\max}$	maximum magnitude of apparent power of line ij (MVA)
$\lvert V_i\rvert^{\min}$	minimum of the voltage magnitude at bus i (kV)
$\lvert V_i\rvert^{\max}$	maximum of the voltage magnitude at bus i (kV)
Y_{ij}	admittance of line ij: ($Y_{ij} = G_{ij} + jB_{ij}$) (℧)
$\theta_{ij}^{\min}$	minimum value of θ_{ij} (rad)
$\theta_{ij}^{\max}$	maximum value of θ_{ij} (rad)
θ_{ref}	voltage phase angle for the slack bus ($\theta_{\mathrm{ref}} = 0$) (rad)
$\Delta\theta_{ij}$	width of each section in binary expansion discretisation method (rad)

Variables

m_{ijk}	binary variables used in binary expansion
OC	operation cost (\$/h)
PG_i	active power generation at bus i (MW)
P_{ij}	active power flow of line ij (MW)
QG_i	reactive power generation at bus i (MVAr)
Q_{ij}	reactive power flow of line ij (MVAr)
$\lvert V_i\rvert$	voltage magnitude at bus i (kV)
x_{ijk}, y_{ijk}	auxiliary variables in BE method
α_{ij}, β_{ij}, γ_{ij}	continuous variables defined in P_{ij} and Q_{ij} linearisation process
θ_{ij}	voltage angle difference between buses i and j (rad)

1 Introduction

Optimal power flow (OPF) research has had a long history and has been one of the most widely researching subjects since its first development by Carpentier in 1962 [1]. The OPF is defined as a non-linear optimisation problem in which one seeks to minimise (or maximise) a specific objective function subject to the physical and operational constraints of the power system.

In general, the OPF is a large-scale and non-convex optimisation problem which is very difficult to solve. This non-convexity is partly as a result of the non-linearity in active and reactive power equations which raises the probability of existence of local solutions for OPF problems. So far, many solution algorithms and methods have been presented and successfully applied to the OPF problem. These methods include [2] Lambda iteration method, gradient method, Newton's method, linear programming method, interior point method and so on. These methods and relevant references have been appropriately discussed in [2]. Also, many heuristic methods, such as genetic algorithm [3], particle swarm optimisation [4], Tabu search [5], artificial bee colony [6] and etcetera were successfully applied to the OPF problem.

A review of selected OPF literature to 1993 is provided by Momoh *et al.* [7, 8]. None of the methods surveyed in the mentioned papers guarantee a global solution to be found at the presence of local ones. Also Huneault and Galiana [9] proposed a quite good paper which surveys 163 papers in the field of OPF and dispatching. A flexible mixed-integer linear programming (MILP) formulation of the alternating current-OPF (AC-OPF) problem for distribution systems, using linearisation techniques are presented in [10]. Wang *et al.* [11] present a market-based OPF using trust-region-based augmented Lagrangian method and step-controlled primal-dual interior point method. A stochastic

OPF problem with stability constraints is presented by Hamon *et al.* [12]. As wind energy continues to increase, new tools and models are proposed for OPF problems due to the intermittent nature of wind flow [13–15].

In this paper, we present a novel linear model and approximated methodology to solve the AC-OPF problem without losing accuracy. It is to be noted that OPF inherently is a non-linear optimisation problem and hence employing a non-linear solver does not guarantee to find a global optimum solution [16–20]. However, a smart initialisation may help to find a good practical feasible local optimum solution that meets the requirements of system operator. Generally speaking, in solving a non-convex optimisation problem, there will be no guarantee to obtain the global optimum solution and this issue remains correct for the most practical optimisation models in a power system including generation and transmission expansion planning in which the mathematical formulations form a mixed-integer non-linear programming (MINLP) problem. It can be said that MINLP problems are the most difficult optimisation problems to solve and no effective algorithms exist to solve such the complex problems. To avoid any local optimal solutions, we transform the non-linear programming (NLP) problem of OPF into an MILP problem incorporating reactive power as well as voltage of buses using binary expansion (BE) discretisation. The problem can be solved efficaciously to the global optimality by existing optimisation software and algorithms. Finding a good solution technique for the full AC-OPF could potentially save tens of billions of dollars annually [21]. It should be highlighted that there exist other techniques and models which try to convexify or linearise the non-convex optimisation problem of AC-OPF. Nevertheless, these methodologies completely differ from that presented in this paper. In [2], a linear programming method based on sensitivity coefficients is suggested which can easily handle the inequality constraints. P.O'Neill *et al.* [22] argue that the current–voltage (*IV*) formulation and its linear approximations may be easier to solve than the traditional quadratic power flow formulations. In [23, 24], it is suggested solving the dual of an equivalent form of the OPF problem rather than the OPF problem itself. This dual problem is a convex semi-definite program and therefore can be solved efficiently in polynomial time. They show that if there is no duality gap a globally optimal solution to the OPF can be recovered from the SDP dual. As stated in [25] the subject of possible existence of local optima for OPF problem is an important issue which has not been well covered in the literature. A MILP model for the solution of the minimum-losses configuration problem of distribution networks is presented by Borghetti *et al.* [26, 27] considering typical operating constraints and the presence of the embedded generation. Also, it is proposed a mixed-integer conic programming formulation for the minimum loss distribution network reconfiguration problem in [28] which employs a convex representation of the network model. In [29], a flexible optimisation-based framework for intentional islanding is presented in which the approach uses a piecewise linear model of AC power flow, which allows the voltage and reactive power to be considered directly. It is evaluated the accuracy and feasibility of the linearised DC model in [30]. Coffrin *et al.* in [30] also propose three new models to improve the accuracy of the linearised DC model.

The linearisation process in this paper involves many techniques including Taylor series expansion theory, binary expansion discretisation approach, piecewise linear approximation and other simple techniques. The OPF has many applications in power system problems such as generation/transmission expansion planning, reactive power planning (RPP), distribution expansion planning and so on. It might be necessary to employ further linearisation methods in order to apply the presented work to these areas. Reactive power has significant effects in reliable and safe operation of power systems. Since the presented work considers both reactive power and voltage, planners can use the proposed method. However, it should be highlighted that in an optimisation problem for the RPP, bus voltage magnitudes may differ more significantly from 1 p.u. which can affect the accuracy of the proposed method. Hence, to assess the preciseness, feasibility and applicability of the proposed method to this area many numerical studies have to be done.

This paper is organised as follows: Section 2 presents the basic AC-OPF formulation. In Section 3, the proposed linear model is illustrated. Simulation results are given in Section 4. Finally, the concluding remarks are drawn in Section 5.

2 AC optimal power flow

Here the basic standard AC-OPF is formulated in usual polar coordinate as follows

$$\text{Min OC} = \sum_{i\in\Omega b} a_i \text{PG}_i^2 + b_i \text{PG}_i + c_i \tag{1}$$

subject to

$$\text{PG}_i - \text{PD}_i = \sum_{j\in\Omega b} P_{ij} \tag{2}$$

$$QG_i - QD_i = \sum_{j\in\Omega b} Q_{ij} \tag{3}$$

$$P_{ij} = \left|V_i\right|^2 G_{ij} - \left|V_i\right|\left|V_j\right|\left(G_{ij}\cos\theta_{ij} + B_{ij}\sin\theta_{ij}\right) \tag{4}$$

$$Q_{ij} = -\left|V_i\right|^2\left(B_{ij} + B_{ij}^{\text{sh}}\right) - \left|V_i\right|\left|V_j\right|\left(G_{ij}\sin\theta_{ij} - B_{ij}\cos\theta_{ij}\right) \tag{5}$$

$$\text{PG}_i^{\min} \le \text{PG}_i \le \text{PG}_i^{\max} \tag{6}$$

$$QG_i^{\min} \le QG_i \le QG_i^{\max} \tag{7}$$

$$\left(P_{ij}\right)^2 + \left(Q_{ij}\right)^2 \le \left(\left|S_{ij}\right|^{\max}\right)^2 \tag{8}$$

$$\left|V_i\right|^{\min} \le \left|V_i\right| \le \left|V_i\right|^{\max} \tag{9}$$

$$\theta_{ij}^{\min} \le \theta_{ij} \le \theta_{ij}^{\max} \tag{10}$$

Equation (1) shows the objective function in which operation cost curves are considered to be quadratic curves. Constraints (2) and (3) enforce the active and reactive power balances at each bus, respectively. Constraints (4) and (5) represent the active and reactive power flows in line *ij*, respectively. Generation limits of each generator are shown by (6) and (7). Equation (8) imposes the thermal limit of transmission lines. Equation (9) maintains the bus voltages within their permissible limits. Constraint (10) shows the angles limitation between nodes. All equations are valid $\forall i, j \in \Omega b$.

The above formulated problem is an NLP problem due to the non-linearity in (4), (5) and (8). It should be noted that the obtained solution for the model (1)–(9) is a local optimum solution rather than a global one.

3 Proposed linear approximated AC-OPF

In this section, the linear AC OPF (LAC-OPF) is formulated. It is to be noted that in [31] a piecewise linear upper approximation is presented which is used here to linearise operation cost of (1). To maintain the system security, it is assumed that θ_{ij} for each bus *i* and *j* which are connected by line *ij* is small enough and voltage magnitude is about 1 p.u. for all buses. These assumptions are practically true under normal operating condition [32]. On the basis of these assumptions, it is proposed to rewrite (4) by replacing sine and cosine functions with their Taylor series expansion about zero and also by substituting quadratic function of $|V_i|^2$ and two-variable function of $|V_i||V_j|$ with their Taylor series

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expansion about 1 as below

$$P_{ij} \cong (2|V_i| - 1)G_{ij} - \left(\underbrace{|V_i| + |V_j| - 1}_{\gamma_{ij}}\right)\left(G_{ij}\left(1 - \frac{\theta_{ij}^2}{2}\right) + B_{ij}\theta_{ij}\right) \tag{11}$$

Assuming $\alpha_{ij} = \gamma_{ij}\theta_{ij}$ and $\beta_{ij} = \gamma_{ij}\theta_{ij}^2$ we obtain

$$\begin{aligned} P_{ij} &\cong (2|V_i| - 1)G_{ij} - \gamma_{ij}G_{ij} + G_{ij}\beta_{ij}/2 - B_{ij}\alpha_{ij} \\ &= G_{ij}\left(|V_i| - |V_j| + \beta_{ij}/2\right) - B_{ij}\alpha_{ij} \end{aligned} \tag{12}$$

Equation (12) is linear with respect to the variables of $|V_i|$, β_{ij} and α_{ij}. However, α_{ij} and β_{ij} are non-linear. To linearise α_{ij} which is the product of two continuous variables, we use the binary expansion approach as proposed in [18]. Effectiveness of BE method was shown in the aforementioned paper for strategic bidding approaches. The basic idea is to approximate the continuous decision variables of θ_{ij} by a set of discrete values as follows

$$\theta_{ij} = \theta_{ij}^{\min} + \Delta\theta_{ij}\sum_{k=0}^{k_1} 2^k m_{ijk} \tag{13}$$

where $\Delta\theta_{ij} = (\theta_{ij}^{\max} - \theta_{ij}^{\min}/2^{k_1})$.

By multiplying the both sides of (13) by variable γ_{ij} it is obtained

$$\alpha_{ij} = \gamma_{ij}\theta_{ij}^{\min} + \Delta\theta_{ij}\sum_{k=0}^{k_1} 2^k x_{ijk} \tag{14}$$

where $x_{ijk} = \gamma_{ij}m_{ijk}$. To linearise x_{ijk} which is a bilinear product of a binary variable (m_{ijk}) and a continuous variable (γ_{ij}), we use the two following equations

$$0 \le \gamma_{ij} - x_{ijk} \le \left(1 - m_{ijk}\right)\mathrm{B}M1_{ij} \tag{15}$$

$$0 \le x_{ijk} \le m_{ijk}\mathrm{B}M1_{ij} \tag{16}$$

To linearise β_{ij} which is the product of three continuous variables, two different methods may be used. One is to approximate θ_{ij}^2 by piecewise linear modelling and then using BE approach. However, this approach introduces new variables and equations which make the problem very large to handle. We propose a linearisation scheme as follows

$$\beta_{ij} = \gamma_{ij}\theta_{ij}^2 = \alpha_{ij}\theta_{ij} = \alpha_{ij}\theta_{ij}^{\min} + \Delta\theta_{ij}\sum_{k=0}^{k_1} 2^k y_{ijk} \tag{17}$$

$$-\left(1 - m_{ijk}\right)BM2_{ij} \le \alpha_{ij} - y_{ijk} \le \left(1 - m_{ijk}\right)BM2_{ij} \tag{18}$$

$$-m_{ijk}BM2_{ij} \le y_{ijk} \le m_{ijk}BM2_{ij} \tag{19}$$

where $y_{ijk} = \alpha_{ij}m_{ijk}$.

The advantage of this method is that, fortunately, it is only required to discretise θ_{ij} when linearising both α_{ij} and β_{ij}. It is to be noted that in BE approach it is needed to define $k_1 + 1$ new binary variables for each discretised variable.

In a similar way, for reactive power one can obtains

$$Q_{ij} \cong -B_{ij}\left(|V_i| - |V_j| + \frac{\beta_{ij}}{2}\right) - G_{ij}\alpha_{ij} - B_{ij}^{\mathrm{sh}}(2|V_i| - 1) \tag{20}$$

The last step is to linearise constraints (8). It should be noted that constraint functions of (8) are convex and the key concern in mathematical programming is convexification not linearisation. However, in order to generate a linear programming model, it is proposed to employ piecewise linear formulations to linearise (8). This allows the use of high performance, efficient and reliable algorithms and solvers so as to solve the resulting MILP problems.

In the P–Q plane, (8) represents a circle with radius $|S_{ij}|^{\max}$. It is proposed to use piecewise linear modelling as depicted in Fig. 1. Indeed, the circle is approximated by an n-sided convex regular polygon. A similar concept for approximation of a circle using polygonal inner approximation has been presented by Ferreira *et al.* [10] and Hamon *et al.* [12]. Therefore, the non-linear equations of (8) are transformed into n linear equations as follows

$$\begin{aligned} &\left(\sin\left(\frac{360^\circ l}{n}\right) - \sin\left(\frac{360^\circ}{n}(l-1)\right)\right)P_{ij} \\ &- \left(\cos\left(\frac{360^\circ l}{n}\right) - \cos\left(\frac{360^\circ}{n}(l-1)\right)\right)Q_{ij} \\ &- |S_{ij}|^{\max} \times \sin\left(\frac{360^\circ}{n}\right) \le 0 \end{aligned} \tag{21}$$

Linear constraints (20) hold for $l = 1, 2, \ldots, n$. The higher the number of sides (n), the more precise the solution is, but at the expense of more computational burden.

The full AC-OPF problem is now completely a linear programming problem which can be solved efficaciously by existing algorithms and solvers ensuring that the global optimal solution is found.

It should be emphasised here that objective function (1) consists of only the power system operation cost in a quadratic format. However, other objective functions such as power losses, load shedding cost, security cost or any other objectives including reactive power or voltage of buses – which are neglected in a DC-OPF approach – can be added to the problem, if necessary. Therefore, it might be required to employ further linearisation techniques to reach to a linear mathematical problem. It is also to be noted that the proposed method guarantees convergence to global optimality in the neighbourhood of $|V_i| = 1$ p.u. and $\theta_{ij} = 0$. However, as the results show, this is completely accepted from the engineering practice point of view not from the viewpoint of mathematics. Indeed, the global solution of the approximated problem is efficiently near the global solution of the exact problem. In the following section, it is applied the presented work to some power systems to show the effectiveness and applicability of the proposed model. As it will be seen shortly the results are promising and show the feasibility and capabilities of the proposed model.

4 Illustrative examples

This section is devoted to case studies and simulation results which are used to demonstrate the proposed method. Different cases are used to show the applicability and accuracy of the method.

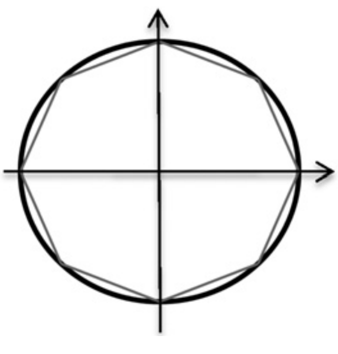

Fig. 1 *Piecewise linear approximation of a circle*

In all simulations, general algebraic modelling system (GAMS) is used as a high-level modelling system for optimisation problems which consists of integrated high-performance solvers [33]. For NLP problems CONOPT is considered as a solver whereas for MILP problems CPLEX are considered as solvers. The relative optimality gap tolerance was set to 0.01%, which results in high accuracy solutions. The GAMS code is run on a computer with Intel Core 2 Duo CPU clocking at 2.00 GHz with 1.00 GB of RAM with Windows 7 as operating system. The presented methodology for OPF is applied to a 2 and 3 bus power system and standard IEEE benchmark system with 24 and 118 buses.

4.1 Two-bus simple power system

Fig. 2 shows a simple two-bus power system. System data can be found in Table 1 in which active powers are in MW and reactive powers are in MVAr. Admittance of lines 1 and 2 is considered to be $Y_{12} = G_{12} + jB_{12} = 15 + j(-60)$ in per unit (p.u.). Maximum and minimum magnitudes of voltage are assumed to be 1.05 and 0.95 p.u. Bus 1 is the slack bus. Total production cost is considered to be the objective function.

For this simple power system, running full AC-OPF with CONOPT under GAMS will lead to a feasible solution. Objective function will be 7404 $/h. Active power generations at buses 1 and 2 are 160 and 140.1 MW, respectively. Reactive power generations are 20 and 40.5 MVAr, respectively. Voltage magnitudes are 1.05 and 1.048 p.u. at buses 1 and 2, respectively.

If lower and upper levels for voltage magnitudes are changed to 0.98 and 1.00 p.u., the same results for the objective function and active power generation are obtained while different voltage profiles are observed (1 and 0.998 p.u. at buses 1 and 2, respectively) which means there is no unique solution for the AC-OPF problem. In other words, different results might be obtained for voltage profile and reactive power flow pattern by different algorithms while trying to solve the same OPF problems in which the objective functions are only related to the cost of active power production. This, however, is due to the existence of multiple solutions for load flow problems which is reported in [25]. In this paper, it is presented a simple load flow example with multiple solutions.

Now, the OPF problem is solved using the proposed MILP model. The objective function is obtained to be 7404 $/h and active power generations are 160 and 140.1 MW. As it can be seen the same results are obtained comparing with the solution results of original OPF, which proves the effectiveness and precision of the presented linear method. However, the voltage magnitudes will be 0.957 and 0.95 p.u. for buses 1 and 2, respectively, and reactive power generations will be 51 and 9 MVAr for generators 1 and 2, respectively, which show significant changes compared with the previous results. To obtain the unique results for the entire variables in both approaches – i.e. NLP approach and proposed MILP approach –, it is needed to incorporate the reactive power generation into the objective function. In doing so, reactive power generation cost is embedded to the objective function with the cost coefficients equal to one-tenth of the corresponding values provided in Table 1 for active power – i.e. $OC = \sum_{i \in \Omega b} b_i PG_i + 0.1 \times b_i QG_i$ –. However, it is emphasised that this is not based on a real operating system and a precise discussion about reactive power offers from generators can be found in [34, 35] which is beyond the scope of this paper. Nevertheless, our simple assumption is enough to investigate the results. The results for two approaches are given in Table 2. As it can be seen the results for reactive power will be approximately the same. The elapsed time to solve the problem by both NLP approach and also MILP approach is reported to be about 0.5 s.

Fig. 2 *Two-bus simple power system*

Table 1 Generators and loads data for two-bus power system

Bus	Generators				Demand	
	b_i, $/MWh	PG_i^{max}, MW	QG_i^{max}, MVAr	QG_i^{min}, MVAr	PD_i, MW	QD_i, MVAr
1	20	160	60	−30	100	20
2	30	160	60	−30	200	40

Table 2 Solution results for two-bus power system

Method	OC, $/h	Bus	PG_i, MW	QG_i, MVAr	V_i, p.u.
NLP	7527	1	160	60	1.05
		2	140.2	0.7	1.042
MILP	7524	1	160	60	0.991
		2	140.1	0.4	0.982

This simple illustrative example clearly shows that the proposed method to solve OPF problems is so effective and accurate and the results will be verifiable and promising. One of the disadvantages of the proposed method is the number of variables, including binary or continuous variables as well as the number of equations which will increase as the system size gets larger. However, it is believed that with the advent of advanced high-speed micro-processors and large memory, this will not be a problem. Be noted that the time it takes to get to the global optimal solution for some purposes such as planning approaches may not be an important issue. Though, this is the expense of getting to the global optimal solution.

4.2 Three-bus simple power system

A three-node power network as shown in Fig. 3 is chosen to apply the method. This system is adopted from [25]. Network data completely can be found in [25]. As it was demonstrated in the aforementioned paper the network has at least three local solutions. These locally optima solutions can be found by randomly generating different initial points for state variables employing the uniform distribution. The solvers used to solve the NLP problem were IPOPT and SNOPT under GAMS. It is also solved the proposed MILP problem using CPLEX. Results are indicated in Table 3. It is to be noted that the quadratic term in objective function is represented by piecewise linear approximation.

As it can be observed from Table 3 the global minimum of the OPF problem is about 5694. Three other local optima solutions have been reported as well. One should be informed that there might be also some other local solutions. However, the solution of 5694 is certainly the global one, since the presented linear approximated method is also converged to it (5699) in which the little difference is due to the assumptions made to the problem to linearise it. This issue proves that the MILP model is precise and

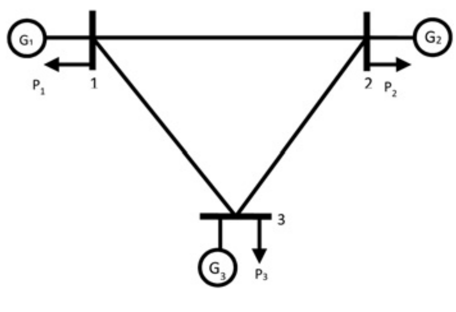

Fig. 3 *Three-bus simple power system*

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Table 3 Solution results for three-bus power system

Method		OC, \$/h	Bus	PG_i, MW	QG_i, MVAr	V_i, p.u.
NLP	1	5694	1	128	42	1.100
			2	188	49	1.100
			3	0	70	1.100
	2	7966	1	181	381	0.900
			2	194	250	0.900
			3	0	403	0.900
	3	9400	1	239	326	0.901
			2	144	231	0.907
			3	0	466	0.903
	4	9693	1	234	444	0.900
			2	165	332	1.100
			3	0	468	1.031
MILP	5699	1	137	38	1.099	
		2	179	52	1.100	
		3	0	72	1.100	

accurate and also it shows the capability of the work to find the global solution. The elapsed time to solve the problem by NLP approach is about 0.5 s and by MILP approach is 2 s.

4.3 24-bus IEEE RTS

As the third example, the well-known 24-bus IEEE RTS is used which is plotted in Fig. 4. This system consists of 10 generators plus a synchronous condenser in buses 14, 17 loads and 38 lines. Area 1 is the 138 kV sub-network and Area 2 is the 230 kV sub-network and 5 tie-lines have connected these two areas. The system data is given in [36].

4.3.1 Linear generation cost curve: The operating costs of the generating units are those provided in [37] and can be found in Table 4. The OPF problem – NLP formulation – is solved using both MATPOWER 4.1 (case 1) [38] and GAMS (case 2). In both cases, the objective function is obtained to be about 56,360 \$/h showing that the same active power dispatch pattern is obtained by means of MATPOWER or GAMS. Generators at buses 1, 2, 7, 13, 15, 16, 18, 21, 22 and 23 produce 192, 192, 156, 49, 66, 155, 400, 400, 169 and 660 MW, respectively. The maximum locational marginal price (LMP) is equal 24.6 \$/MWh and belongs to node 8 while the minimum LMP 21.2 \$/MWh belongs to bus 22. However, the voltage profiles of buses and also reactive power generation are different in these two cases.

Solving linearised OPF by GAMS – third case – results in an objective function with the same amount as mentioned above, however, different profiles for bus voltage and therefore different flow patterns for reactive power are observed. To achieve these results, it is assumed $n = 64$ and $k_1 = 14$. The program coded in GAMS has 13,335 single equations, 5059 single variables and 510 binary variables. Be noted it is only required to discretise θ_{ij} in which ij refers to a transmission line connecting buses i to j. As it can be seen, for this particular case, both NLP and MILP methods are converged to the global optimal solution. As it was seen earlier, this is not going to be general for all cases. Again, incorporating reactive power cost in objective function results in the same outcomes for reactive power dispatch in power system. The elapsed time to solve the problem by NLP approach is about 2 s and by MILP approach is 1709 s.

Other objective functions such as power losses can also be considered. It should be highlighted that active power losses in corridor ij can be calculated as $Ploss_{ij} = P_{ij} + P_{ji}$. If the objective function is considered to be only the operating cost of generators, the active power losses will be 41.2 MW. Considering the active power losses as the objective function reduces the power losses to 27.3 MW. However, the operation cost increases to 57,676 \$/h. These two objective functions are conflicting and it is necessary to employ a multi-objective optimisation approach to obtain Pareto optimal solutions. These results are verified by both NLP approach as well as by MILP approach.

Including shunt admittance in simulation study has little effect on objective function which is defined to be the operation cost. Objective function increases from 56,360 to 57,045 \$/h. The active power flow, also, would be approximately the same as before. However, as it was expected, shunt admittances affect the voltage profile as well as reactive power flow. This issue is also verified for the next numerical study for 118-bus IEEE test system.

It should be emphasised here that one has to be careful about determining parameters n and k_1. A small n imposes more restrictions on transmission capacity and it may make the problem infeasible while a big n would increase the number of equations. Also, the number of binary variables depends on k_1. While a small k_1 might lead to the less efficient solution, a big k_1 would lead to too many binary variables and a larger search space to be explored.

The results are compared for different n and k_1 and are shown in Table 5. We also have compared the time to solve the optimisation problem and also the number of variables/equations. As it can be seen, the time it takes to reach the optimal solution will increase as the number n increases. The number n can be different for various lines. It clearly depends on the operating point of active and reactive powers of each line in the corresponding P–Q plane. For highly congested transmission lines, number n have to be large enough, but for the lines operating far from their limits n can be smaller without losing any generality. Solving OPF using simple DC power flow might be helpful to identify that which lines are more congested. Meanwhile, the engineer's experience would help to select the number n and k_1. Also, it is clear from Table 5 that as the number n reduces, the operation cost is increased due to the commitment of more expensive generating units because of imposing more restrictions on transmission lines capacity.

4.3.2 Quadratic generation cost curve: The operating costs of the generating units are considered to be quadratic curves, i.e. in the form of $F(\mathrm{PG}_i) = a_i\mathrm{PG}_i^2 + b_i\mathrm{PG}_i + c_i$. To employ an MILP approach, the quadratic curves have to be linearised by piecewise linear approximation method. In [31], a piecewise linear upper approximation is presented which is used in this paper. It is assumed $a_i = 0.01b_i$ and $c_i = 100b_i$ in which coefficient b_i is provided in Table 4. In NLP approach the value of 259,405 \$/h is obtained for objective function. Approximating the quadratic curve by piecewise linear method and solving MILP approach results in the value of 260,910 \$/h for objective function. As stated in [31] a piecewise linear lower approximation can also be used in which the piecewise linear approximated operation costs are no larger than the corresponding values calculated from the original quadratic cost curve.

4.4 118-bus IEEE test system

The IEEE 118-bus test system has been also used to apply the presented work. To better compare the results, the original standard AC-OPF problem which was formulated in Section 2 is programmed in GAMS. It is to be noted that the cost function coefficients are taken from [38] in which the quadratic terms are omitted here. For this AC-OPF-based case the total production cost is obtained to be 81,288 \$/h. It takes about 13 s to run the program. The coded program has 42,461 single equations and 28,193 single variables for this case. The same result is approximately obtained by the proposed method using LAC-OPF. In this method, the objective function would be 80,784 \$/h and the written program has 154,393 single equations, 116,837 single variables and 1700 binary variables.

All NLP solvers are converged to 81,288 and no local solutions are found in different iterations. To produce local optima solutions active and reactive power limits are all relaxed as discussed in [25]. In doing so, two local solutions are found which are approximately 34 and 47% higher than the global one.

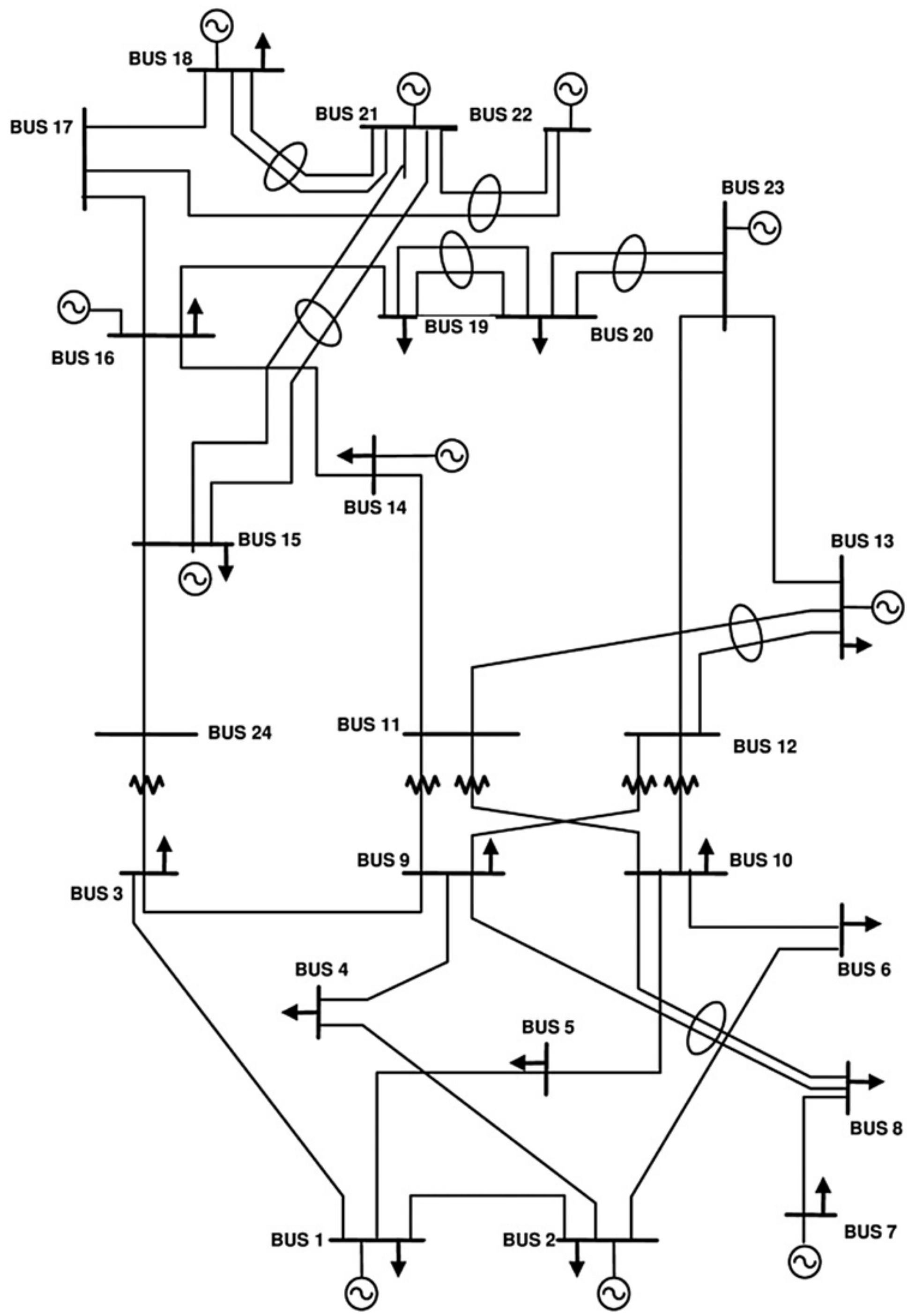

Fig. 4 *24-bus IEEE RTS*

Table 4 Generators and loads data For RTS

Bus	Generators				
	b_i, $/MWh	PG_i^{max}, MW	PG_i^{min}, MW	QG_i^{max}, MVAr	QG_i^{min}, MVAr
1	20.3	192	62.4	80	−50
2	18.5	192	62.4	80	−50
7	24.4	300	75	180	0
13	22.2	591	207	240	0
14	–	–	–	200	−50
15	23.6	215	66.3	110	−50
16	19.1	155	54.3	80	−50
18	16.0	400	100	200	−50
21	20.8	400	100	200	−50
22	21.2	300	60	96	−60
23	16.9	660	248.6	310	−125

Table 5 Results for different values of n and k_1 for RTS

	OC, $/h	Number of single equations/ continuous variables/discrete variables	Elapsed time, s
$k_1 = 7$, $n = 32$	57,901	9255/4345/272	177
$k_1 = 7$, $n = 64$	57,755	11,431/4345/272	259
$k_1 = 14$, $n = 32$	56,932	11,159/5059/510	1174
$k_1 = 14$, $n = 64$	56,360	13,335/5059/510	1709
$k_1 = 20$, $n = 128$	56,360	19,319/5671/714	5503

5 Concluding remarks

In this paper, a new linear AC-OPF model was presented. The linearisation process in this paper involves many techniques including Taylor series expansion theory, binary expansion discretisation approach, piecewise linear approximation and other simple techniques. Theoretical developments of the presented work and its application were comprehensively reported. The presented method provides more precise and real picture of both active and reactive power flows along with the voltage profile of the network. The AC-OPF is an NLP problem that is transformed into an MILP by the proposed method. In doing so, a new linearisation approximated method is presented to transform the non-linear model into a linear one. The formulated problem then can be solved by available commercial and efficient algorithms and software which due to the linearity of the proposed formulated problem, the global solution of the approximated model is guaranteed to be found as shown in the numerical studies. The proposed method guarantees convergence to global optimality in the neighbourhood of $|V_i| = 1$ p.u. and $\theta_{ij} = 0$. However, as the results show, this is completely accepted from the engineering practice point of view. Indeed, the global solution of the approximated problem is efficiently near the global solution of the exact problem. One advantage of this work is that active and reactive power flow patterns and voltage profiles are obtained at the same time. However, one main disadvantage of the method is the exponential growth in the execution time when using CPLEX as solver which employs the branch and bound method to solve the problem. Some of the applications were named for which the applicability of the presented work can be investigated and can be considered as good future researches. Simulation results are presented and thoroughly discussed for small systems as well as power networks of standard IEEE benchmark systems.

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